November, 2001

Supplemental Materials for:
How to Set Minimum Acceptable Bids, with an Application to Real Estate Auctions

by
R. Preston McAfee,
Daniel C. Quan,
and
Daniel R. Vincent[*]

Journal of Industrial Economics,
VOLUME(ISSUE), MONTH, YEAR, pp. XXX-YYY

Proof of Lemma 3 for First Price Auctions: The first order condition determining a bid for a bidder of type *x* is derived as follows. A bidder of type *x* who submits a bid as if he were type *x'* obtains

$$u(x,x') = E_\theta\left[(u(x,\theta) - B(x')(1-\rho(1-F(x'|\theta)))^{n-1} f(x|\theta)\right].$$

Therefore, *B(x)* must solve

$$E_\theta[(u(x,\theta) - B(x)(1-\rho(1-F(x|\theta)))^{n-2}(n-1)\rho f^2(x|\theta)$$
$$-B'(x)(1-\rho(1-F(x|\theta)))^{n-1} f(x|\theta)] = 0.$$

To show (7), first integrate (3) by parts to obtain

$$\text{(A3F)} \quad s = E_\theta\left[\int_{x_r}^{\infty}(u(x,\theta)-B(x))(1-\rho(1-F(x|\theta)))^{n-1} f(x|\theta)dx\right]$$

$$= -E_\theta\left[\int_{x_r}^{\infty}(u(x,\theta)-B(x))(1-\rho(1-F(x|\theta)))^{n-1}\frac{d(1-F(x|\theta)}{dx}dx\right]$$

$$= \begin{array}{l} E_\theta[\int_{x_r}^{\infty}(1-F(x_r|\theta))(u(x_r,\theta)-B(x_r))(1-\rho(1-F(x_r|\theta)))^{n-1}\,dx + \\ \int_{x_r}^{\infty}(1-F(x|\theta))(u_x(x,\theta)-B'(x))(1-\rho(1-F(x|\theta)))^{n-1}\,dx + \\ \int_{x_r}^{\infty}(1-F(x|\theta))(u(x,\theta)-B(x))(n-1)(1-\rho(1-F(x|\theta)))^{n-2}\rho f(x|\theta)\,dx] \end{array}$$

Moreover,

$$E_\theta\,[(1-F(x|\theta))\,\{(u(x,\theta)-B(x))(n-1)(1-\rho(1-F(x|\theta)))^{n-2} f(x|\theta) -$$
$$B'(x)(1-\rho(1-F(x|\theta)))^{n-1}\}]$$

$$= E_\theta\,[\{\frac{1-F(x|\theta)}{f(x|\theta)}\}\,\{(u(x,\theta)-B(x))(n-1)(1-\rho(1-F(x|\theta)))^{n-2} f^2(x|\theta) -$$
$$B'(x)(1-\rho(1-F(x|\theta)))^{n-1} f(x|\theta)\}]$$

$$=E_\theta\left[\left\{\frac{1-F(x|\theta)}{f(x|\theta)}\right\}\left\{u(x,\theta)-B(x)-B'(x)\frac{(1-\rho(1-F(x|\theta)))^{n-1}f(x|\theta)}{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)}\right\}\times\{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)\}\right].$$

Affiliation implies that the first term in curly brackets is non-decreasing. By assumption, u is nondecreasing in θ. Also,

$$-B'(x)\frac{(1-\rho(1-F(x|\theta)))^{n-1}f(x|\theta)}{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)}$$

is increasing in θ since B is increasing in x and $(1-\rho(1-F(x|\theta)))/f(x|\theta)$ is decreasing in θ. (See Equation (1)). Thus,

$$\text{(A4F)}\quad E_\theta\left[\left\{\frac{1-F(x|\theta)}{f(x|\theta)}\right\}\left\{u(x,\theta)-B(x)-B'(x)\frac{(1-\rho(1-F(x|\theta)))^{n-1}f(x|\theta)}{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)}\right\}\times\{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)\}\right]$$

$$\geq E_\theta\left[\left\{\frac{1-F(x|\theta)}{f(x|\theta)}\right\}\{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)\}\right]\times$$
$$E_\theta\left[\left\{u(x,\theta)-B(x)-B'(x)\frac{(1-\rho(1-F(x|\theta)))^{n-1}f(x|\theta)}{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)}\right\}\{(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f^2(x|\theta)\}\right]$$

the second term of which is zero by the first order condition determining the bid function.

Integrating (5) by parts,

$$\Psi = \begin{array}{l} E_\theta\left[\sigma(\theta)-(u(x,\theta)-\sigma(\theta))\left(1-(1-\rho(1-F(x|\theta)))^n\right)\Big|_{x_r}^{\infty}\right. \\ \left.+\int_{x_r}^{\infty}u_x(x,\theta)\left(1-(1-\rho(1-F(x|\theta)))^n\right)dx - n\rho s\right] \end{array}$$

$$= \begin{array}{l} E_\theta\left[\sigma(\theta)-(u(x_r,\theta)-\sigma(\theta))\left(1-(1-\rho(1-F(x_r|\theta)))^n\right)\right. \\ \left.+\int_{x_r}^{\infty}u_x(x,\theta)\left(1-(1-\rho(1-F(x|\theta)))^n\right)dx - n\rho s\right]. \end{array}$$

Therefore, differentiating with respect to ρ,

$$\frac{\partial \Psi}{\partial \rho} = \begin{array}{l} nE_\theta\,[(u(x_r,\theta) - \sigma(\theta))(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1} + \\ \int_{x_r}^{\infty} u_x(x,\theta)(1-\rho(1-F(x|\theta)))^{n-1}(1-F(x|\theta))\,dx - s] \end{array}$$

$$\overset{(A3F)}{=} \quad nE_\theta\,[(r-\sigma(\theta))(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1} +$$

$$- \int_{x_r}^{\infty}(1-F(x|\theta))(u(x,\theta) - B(x))(n-1)(1-\rho(1-F(x|\theta)))^{n-2}\rho f(x|\theta)\,dx\,]$$

$$\overset{(A4F)}{\leq} \quad nE_\theta\,[(r-\sigma(\theta))(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}\,].$$

where the fact that $B(x_r)=r$ for first price auctions is used.

■

Proof of Lemma 3 for English Auctions: This proof is more complex because, as the bidding proceeds, the bidder observes who is participating in the auction. We assume that once bidders have made their decisions whether to purchase a signal and whether to bid at the auction, each bidder who does show up, observes how many bidders, say $i+1$, are competing. From that point onward, they utilize the Milgrom and Weber (1982) equilibrium strategies for an English auction with $i+1$ bidders. Fix bidder one and let the signals of the other n bidders be ordered as $y_1 \geq y_2 \geq \ldots \geq y_i$. Define $\bar{y}_j = (y_2, y_3, \ldots, y_{j+1})$. Thus, $\bar{y}_{i-1}$ is the ordered vector of the lowest signals except bidder one and the top rival given $i+1$ bidders arrive and bid at the auction.

Define

$$B^i(x;\bar{y}_{i-1}) = E_\theta[u(x,\theta)|x, y_1 = x, \bar{y}_{i-1}]$$

to be the equilibrium bidding function when i bidders compete with bidder one and conditional on the i-1 values of the lowest competing bidders being revealed by their dropout strategy. The bid function is the highest value that a bidder with signal x will bid given he knows i-1 lowest signals. Equation (2) yields

$$s = E_\theta \int_{x_r}^{\infty} f(x|\theta)[\,u(x,\theta)(1-\rho(1-F(x|\theta)))^{n-1} - r(1-\rho(1-F(x_r|\theta)))^{n-1}$$

$$- \int_{x_r}^{x}(n-1) - B^1(y_1)(1-\rho(1-F(x_r|\theta)))^{n-2}\rho f(y_1|\theta)dy_1$$

$$- \int_{x_r}^{x}\int_{x_r}^{y_1} \frac{(n-1)!}{(n-3)!} B^2(y_1;\bar{y}_1)(1-\rho(1-F(x_r|\theta)))^{n-3}\rho^2 f(y_2|\theta)dy_2 f(y_1|\theta)dy_1$$

- ...

$$- \int_{x_r}^{x}\int_{x_r}^{y_1}\dots\int_{x_r}^{y_{i-1}} \frac{(n-1)!}{(n-i-1)!} B^i(y_1;\bar{y}_{i-1})(1-\rho(1-F(x_r|\theta)))^{n-i-1}\rho^i \ \Pi_{j=i}^{1} f(y_j|\theta)dy_j$$

-...

$$- \int_{x_r}^{x}\int_{x_r}^{y_1}\dots\int_{x_r}^{y_{n-2}} (n-1)! B^{n-1}(y_1;\bar{y}_{n-2})\rho^{n-1} f(y_{n-1}|\theta)dy_{n-1}\dots f(y_2|\theta)dy_2 f(y_1|\theta)dy_1]dx.$$

(The bar over the y denotes a vector of signals.) Integrate all but the first line by parts to obtain:

(A3E) $$s = E_\theta\Bigg[\int_{x_r}^{\infty} u(x,\theta)(1-\rho(1-F(x|\theta)))^{n-1} f(x|\theta)dx - r(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}dx$$

$$- \int_{x_r}^{\infty} (1-F(x|\theta))\rho f(x|\theta)\Big\{(n-1)B^1(x)(1-\rho(1-F(x_r|\theta)))^{n-2}$$

$$- \int_{x_r}^{x} \frac{(n-1)!}{(n-3)!} B^2(x;\bar{y}_1)(1-\rho(1-F(x_r|\theta)))^{n-3}\rho f(y_2|\theta)dy_2$$

-...

$$- \int_{x_r}^{x}\dots\int_{x_r}^{y_{i-1}} \frac{(n-1)!}{(n-3)!} B^i(x;\bar{y}_{i-1})(1-\rho(1-F(x_r|\theta)))^{n-i-1}\rho^{i-1} f(y_i|\theta)dy_i\dots f(y_2|\theta)dy_2$$

-...

$$- \int_{x_r}^{x}\dots\int_{x_r}^{y_{n-2}} (n-1)! B^{n-1}(x;\bar{y}_{n-2})\rho^{n-2} f(y_{n-1}|\theta)dy_{n-1}\dots f(y_2|\theta)dy_2\Big\}dx\Bigg].$$

Define the expectations operator,

$$\hat{E}^{i}_{x\bar{y}_{i-1}}(\bullet) = \frac{E_\theta\left[(\bullet)(1-\rho(1-F(x_r|\theta)))^{n-i-1} f(y_i|\theta) f(y_{i-1}|\theta)\dots f(y_2|\theta) f(x|\theta)^2\right]}{E_\theta\left[(1-\rho(1-F(x_r|\theta)))^{n-i-1} f(y_i|\theta) f(y_{i-1}|\theta)\dots f(y_2|\theta) f(x|\theta)^2\right]}.$$

For all *i,* and for any revealed *i-1* signals,

(A4E) $$E_\theta\left[(1-F(x|\theta))(u(x,\theta) - B^i(x;\bar{y}_{i-1}))(n-1)(1-\rho(1-F(x_r|\theta)))^{n-i-1} f(y_i|\theta)\dots f(y_2|\theta) f(x|\theta)\right]$$

$$= \frac{\hat{E}_{x,\bar{y}_{i-1}}\left[\frac{1-F(x|\theta)}{f(x|\theta)}(u(x,\theta) - B^i(x;\bar{y}_{i-1}))\right] \times}{E_\theta\left[(1-\rho(1-F(x_r|\theta)))^{n-i-1} f(y_i|\theta) f(y_{i-1}|\theta) \ldots f(y_2|\theta) f(x|\theta)^2\right]}$$

$$\geq \frac{\hat{E}_{x,\bar{y}_{i-1}}\left[\frac{1-F(x|\theta)}{f(x|\theta)}\right]\hat{E}_{x,\bar{y}_{i-1}}\left[(u(x,\theta) - B^i(x;\bar{y}_{i-1}))\right] \times}{E_\theta\left[(1-\rho(1-F(x_r|\theta)))^{n-i-1} f(y_i|\theta) f(y_{i-1}|\theta) \ldots f(y_2|\theta) f(x|\theta)^2\right]}$$

$= 0.$

The last equality is by definition of the bidding functions. The inequality is an implication of affiliation. Seller revenues are given by

$$\Psi = E_\theta\left[\sigma(\theta)(1-\rho(1-F(x_r|\theta)))^n + \int_{x_r}^{\infty}(u(x,\theta))\, n(1-\rho(1-F(x|\theta)))^{n-1}\rho f(x|\theta)\, dx - n\rho s\right].$$

Differentiating with respect to ρ yields

$$\frac{\partial \Psi}{\partial \rho} = -\ E_\theta\left[(1-F(x_r|\theta)) n\sigma(\theta)(1-\rho(1-F(x_r|\theta)))^{n-1}\right]$$

$$+E_\theta\left[\int_{x_r}^{\infty} u(x,\theta)\, n(1-\rho(1-F(x|\theta)))^{n-1} f(x|\theta)\, dx\right]$$

$$-E_\theta\left[\int_{x_r}^{\infty} u(x,\theta)\, n(n-1)(1-F(x|\theta))(1-\rho(1-F(x|\theta)))^{n-2}\rho f(x|\theta)\, dx + n s\right]$$

$$\overset{(A3E)}{=} nE_\theta\left[(1-F(x_r|\theta))(r-\sigma(\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}\right]$$

$$-\, nE_\theta\left[\int_{x_r}^{\infty}(1-F(x|\theta))u(x,\theta)(n-1)(1-\rho(1-F(x|\theta)))^{n-2}\rho f(x|\theta)\, dx\right]$$

$$+nE_\theta\left[\int_{x_r}^{\infty}(1-F(x|\theta))\rho f(x|\theta)\Big\{(n-1)(B^1(x))(1-\rho(1-F(x_r|\theta)))^{n-2}\right.$$

$$+\int_{x_r}^{x}\frac{(n-1)!}{(n-3)!}B^2(x;y_2)(1-\rho(1-F(x_r|\theta)))^{n-3}\rho f(y_2|\theta)dy_2$$

$$+ ....$$

$$+\int_{x_r}^{x}...\int_{x_r}^{y_{i-1}}\frac{(n-1)!}{(n-3)!}B^i(x;\overline{y}_{i-1})(1-\rho(1-F(x_r|\theta)))^{n-i-1}\rho^{i-1}f(y_i|\theta)dy_i...f(y_2|\theta)dy_2$$

$$+....$$

$$\left.+\int_{x_r}^{x}...\int_{x_r}^{y_{n-2}}(n-1)!B^{n-1}(x;\overline{y}_{n-2})\rho^{n-2}f(y_{n-1}|\theta)dy_{n-1}...f(y_2|\theta)dy_2\Big\}dx\right].$$

Observe that

$$(1-\rho(1-F(x|\theta)))^{n-2} = (1-\rho(1-F(x_r|\theta)))^{n-2}+(1-\rho(1-F(x|\theta)))^{n-2}-(1-\rho(1-F(x_r|\theta)))^{n-2}.$$

For any y_i and any j, $n\text{-}2 \geq j \geq 1$, we have

$$(1-\rho(1-F(y_i|\theta)))^j-(1-\rho(1-F(x_r|\theta)))^j = \int_{x_r}^{y_i}j(1-\rho(1-F(y_{i+1}|\theta)))^{j-1}\rho f(y_{i+1}|\theta)dy_{i+1}$$

We apply this relation iteratively, to decompose the expression

$$-\ nE_\theta\left[\int_{x_r}^{\infty}(1-F(x|\theta))u(x,\theta)(n-1)(1-\rho(1-F(x|\theta)))^{n-2}\rho f(x|\theta)\,dx\right]$$

yielding

$$\frac{\partial\Psi}{\partial\rho} = nE_\theta\Big[(1-F(x_r|\theta)(r-\sigma(\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}\Big]$$

$$-n\ E_\theta\left[\int_{x_r}^{\infty}(1-F(x|\theta))\rho f(x|\theta)\times\right.$$

$$\Big\{(u(x,\theta)-B^1(x))(n-1)(1-\rho(1-F(x_r|\theta)))^{n-2}$$

$$+\int_{x_r}^{x}\frac{(n-1)!}{(n-3)!}(u(x,\theta)-B^2(x;y_2))(1-\rho(1-F(x_r\,|\,\theta)))^{n-3}\rho f(y_2\,|\,\theta)dy_2$$

$+ ....$

$$+\int_{x_r}^{x}\cdots\int_{x_r}^{y_{i-1}}\frac{(n-1)!}{(n-3)!}(u(x,\theta)-B^i(x;\overline{y}_{i-1}))(1-\rho(1-F(x_r\,|\,\theta)))^{n-i-1}\rho^{i-1}f(y_i\,|\,\theta)dy_i\ldots f(y_2\,|\,\theta)dy_2$$

$+....$

$$+\int_{x_r}^{x}\cdots\int_{x_r}^{y_{n-2}}(n-1)!(u(x,\theta)-B^{n-1}(x;\overline{y}_{n-2}))\rho^{n-2}f(y_{n-1}\,|\,\theta)dy_{n-1}\ldots f(y_2\,|\,\theta)dy_2\Big\}dx\Big].$$

$$\overset{(A4E)}{\leq}\quad nE_\theta\left[(r-\sigma(\theta))(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}\right].\qquad\blacksquare$$