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The Evolutionary Stability of Auctions over Bargaining*

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This paper considers equilibrium in transaction mechanisms. In an environment with homogeneous buyers and sellers, which eliminates the advantage auctions possess of matching buyers and sellers, both auctions and bargaining are equilibria. However, only auctions are evolutionarily stable. This identifies a new advantage of auctions over bargaining, arising from the division of the gains from trade. *Journal of Economic Literature* Classification Numbers: C78, C73, D44.

1. INTRODUCTION

Beginning with Diamond (1971), a great deal of attention has been paid to the microstructure of markets and trading institutions. The literature divides naturally into four categories. The first category focuses on behavior in one institution, e.g., Rubinstein and Wolinsky (1985). The second type restricts attention to different types of auctions and compares them in an attempt to reveal the intuition for why certain forms of auction are more frequently used in the real world than the others (Milgrom and Weber, 1982; Milgrom, 1987). The third type compares different institutions to determine the structural advantages of certain institutions (Arnold and Lippman, 1995; De Vany, 1987; Ehrman and Peters, 1993; Wang, 1993, 1995). The fourth type explores endogenous equilibrium institutions (McAfee, 1993; Peters, 1994). This paper belongs to the third group. We focus on auctions and bargaining as rival institutions. But, like

* We thank two anonymous referees for thoughtful comments, including providing an intuition to Lemma 2, which dramatically simplifies the proof of Lemma 2a, and for pointing out a difficulty with the argument concerning the step size of the evolutionary process. We are responsible for all errors. McAfee thanks MIT for its hospitality.

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the fourth category, we consider an explicit selection mechanism (evolutionary stability) for the institution.

Auctions and bargaining are two very common trading institutions. Construction contracts, works of art, and fine wines are a few examples of goods and services sold through auctions. Houses and cars, on the other hand, are usually sold through bargaining. In many situations, the goods sold through either bargaining or auction institutions have similar properties: they are unique, expensive, and with uncertain equilibrium prices.

In many circumstances, auctions are superior to bargaining. By using a model of monopoly with random matching heterogeneous buyers and the possibility to resell, Milgrom (1987) pointed out that auctions often lead to an efficient and stable outcome. McAfee and McMillan (1988) showed that a combination of reservation-price search and auction is, with costly communication, a monopsonist's optimal procurement mechanism when the potential sellers have different production costs. In a more complicated, "near perfect competition" situation, McAfee (1993) demonstrated that sellers holding identical auctions and buyers randomizing over the sellers they visit comprises an equilibrium, when all mechanisms are available to sellers (see also Peters, 1994).

The intuition that auctions have an inherent advantage over bargaining mechanisms with random matching among the players is straightforward. Auctions have the ability to discriminate among buyers and choose the highest value buyer (McAfee and McMillan, 1987; Milgrom, 1987).

In the absence of this advantage (e.g., homogeneous environments in which buyers and sellers are all of one type), it is unclear whether auctions remain superior to bargaining. It would appear that either institution could arise as an equilibrium, since if all buyers bargain, every seller wishes to bargain, and conversely.

This paper examines auctions and bargaining in homogeneous environments, using an evolutionary framework. We find that although both auctions and bargaining are equilibrium institutions, bargaining is unstable under a wide class of evolutionary dynamics, and thus auctions tend to be selected as the only stable equilibrium mechanism. The result suggests that auctions have an advantage over bargaining mechanisms with random matching even without the ability of sellers to discriminate among buyers and choose the highest value buyer.

The advantage auctions possess over bargaining concerns the division of the surplus between buyers and sellers. An increase in the ratio of buyers to sellers will disadvantage buyers in both transaction mechanisms. However, it will disadvantage buyers relatively less in auctions as compared to bargaining in the circumstances when buyers prefer auctions over bargaining, in a sense made precise below. The consequence of this differential distribution of the gains from trade is circumstances (the ratio of buyers to sellers in auctions and bargaining, respectively) where both buyers and sellers prefer auctions to bargaining. The existence of such circumstances leads to the global stability of auctions.

The remaining sections of the paper are organized as follows. In the next section, we establish a model in which sellers can either hold auctions or bargain and buyers participate in either auctions or bargaining. In Section 3, we show that all agents choosing auctions or all agents bargaining are the only two steady state equilibria in almost all circumstances. Then, in Section 4, we prove that the unique globally stable equilibrium is for all agents to choose auctions. The conclusion is in Section 5.

2. THE MODEL

There are two types of agents in the model, buyers and sellers. Each seller has one unit of an indivisible good for sale and each buyer seeks to buy exactly one unit of this good. All sellers are homogeneous in the sense that the goods they sell, and the sellers' values of the goods, are identical. All buyers are also homogeneous in the sense that the consumption values of the good to the buyers are the same, and normalized to unity. Each seller's use value is set to zero; the seller's value of not selling, which is the discounted value of being a seller in the next period, will arise endogenously.

Two separate markets exist simultaneously. One is an auction market and the other is a bargaining market. Each seller can choose to either hold auctions or to bargain, and simultaneously each buyer can choose to either attend an auction or to match with a seller who bargains. It is assumed that once an agent enters a market, the cost of transferring to the other is prohibitive. Thus, only new agents change institutions.[1]

Time is discrete, $t = 1, 2, 3, \ldots$. In each period t, there are $\theta_t N_t + N_t$ agents in the market, where N_t is a very large integer and $0 < \theta_t < +\infty$. Among them, $\theta_t N_t$ are buyers and N_t are sellers. We denote the proportions of buyers and sellers in the bargaining market at time t by x_t and y_t. Consequently, the proportions of buyers and sellers in the auction market at time t are $1 - x_t$ and $1 - y_t$.

At the beginning of each period, all buyers in each market are randomly spread over the sellers, so that a buyer can match with at most one seller while a seller may be visited by multiple buyers.[2] The number of buyers visiting each seller in either the bargaining or auction markets is a binomial random variable, with parameters $\theta_t N_t x_t$ and $1/N_t y_t$ for the bargaining market, and $\theta_t N_t(1 - x_t)$ and

[1] Fixing the behavior of old agents simplifies the exposition. For the evolutionary dynamic to operate, it is necessary that a fraction of agents do not switch to the market offering higher utility. The analysis will presume that only an insignificant fraction of the total agents switch to the market offering higher utility. This may require that some entering agents actually follow the behavior of their predecessors.

[2] If the sellers have fixed locations and each buyer chooses a seller to visit simultaneously, this model would arise in a symmetric equilibrium.

$1/N_t(1 - y_t)$ for the auction market. Since N_t is a very large number, these distributions can be well approximated by Poisson distributions with parameters $\theta_t x_t/y_t$ and $\theta_t(1-x_t)/(1-y_t)$ respectively which hold exactly in the limit as $N_t \to \infty$. The successful buyers and the corresponding prices of trade are determined through either auction or bargaining institutions. Once a trade is completed, the buyer and the seller involved in the trade leave the market. Afterward a fraction γ of the remaining buyers and sellers is terminated exogenously and the rest of the agents will wait for an opportunity next period. At the end of the period, new buyers and sellers come and join either the auction or the bargaining market based on their market evaluations. It is assumed that buyers and sellers have the same discount factor δ over the value one period ahead.

Despite having the same structure of the matching process, auction and bargaining institutions produce different expected utilities. The distributions of the stock buyers and sellers in the markets determine which market produces higher utilities.

2.1. *The Bargaining Market*

In the bargaining market, we assume that each seller picks a buyer randomly from the available buyers she matches if she happens to have multiple buyers. Since the number of buyers for a given seller is a Poisson random variable with parameter $\theta_t x_t/y_t$, i.e.,

$$P\{K = k\} = \frac{e^{-\theta_t x_t/y_t}(\theta_t x_t/y_t)^k}{k!} \qquad (k = 0, 1, 2, \ldots),$$

the probability that a seller has a bargaining partner equals $1 - e^{-\theta_t x_t/y_t}$. A buyer may not have the chance to bargain even if he runs into a seller, depending upon the existence of a competitor. If there are k ($k \geq 1$) other buyers visiting the same seller, the probability that he is chosen by the seller is $1/(k+1)$. Consequently, the probability that a buyer can actually find a seller to bargain with is

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \frac{e^{-\theta_t x_t/y_t}(\theta_t x_t/y_t)^k}{k!} = \frac{y_t(1 - e^{-\theta_t x_t/y_t})}{\theta_t x_t}.$$

We use U_t^B and V_t^B to denote the expected utilities of the buyers and the sellers in the bargaining market. There are two types of discounting: that arising from the threat of termination (probability γ) and pure time preference. Both are included in the discount factor δ. Thus, the gain from trade in the bargaining market is $G_t^B = 1 - \delta U_{t+1}^B - \delta V_{t+1}^B$, where δU_{t+1}^B and δV_{t+1}^B are the values of not trading for the buyers and sellers, respectively. To concentrate on the subject of interest and make the bargaining process as simple as possible, we assume that the gain from trade will be split evenly between the buyer and the seller in bargaining, which is the outcome under Nash bargaining.[3] Consequently, the expected price of one unit of the good is

$$P_t = \frac{G_t^B}{2} + \delta V_{t+1}^B = 1 - \frac{G_t^B}{2} - \delta U_{t+1}^B.$$

When a buyer gets an opportunity to bargain with a seller, he will gain $1 - P_t$. Otherwise he will wait with an expectation of δU_{t+1}^B. The expected utility of a buyer in the bargaining market, therefore, satisfies the following dynamic equation:

$$\begin{aligned} U_t^B &= \frac{y_t(1 - e^{-\theta_t x_t / y_t})}{\theta_t x_t}(1 - P_t) + \left(1 - \frac{y_t(1 - e^{-\theta_t x_t / y_t})}{\theta_t x_t}\right)\delta U_{t+1}^B \\ &= \frac{G_t^B}{2}\frac{y_t(1 - e^{-\theta_t x_t / y_t})}{\theta_t x_t} + \delta U_{t+1}^B. \end{aligned} \tag{1}$$

Similarly, the expected utility of a seller in the bargaining market can be expressed as

$$V_t^B = (1 - e^{-\theta_t x_t / y_t})P_t + e^{-\theta_t x_t / y_t}\delta V_{t+1}^B = \frac{G_t^B}{2}(1 - e^{-\theta_t x_t / y_t}) + \delta V_{t+1}^B. \tag{2}$$

2.2. *The Auction Market*

In the auction market, we assume that each seller is committed to sell the good to the buyer with the highest bid, so long as the bid is not lower than the reservation value.[4] If there is a tie at the highest bid, the seller will break the tie at random.

The number of buyers participating in an auction follows Poisson distribution with parameter $\theta_t(1 - x_t)/(1 - y_t)$, i.e.,

$$P\{K = k\} = \frac{e^{-\theta_t(1-x_t)/1-y_t}\left(\frac{\theta_t(1-x_t)}{1-y_t}\right)^k}{k!} \qquad (k = 0, 1, 2, \ldots).$$

A buyer at a particular auction faces two possibilities. The first is that he is the only buyer in the auction. In this case, the optimal strategy for the buyer is to bid at the reservation value of the seller. The second possibility is that there is at

[3] See Rubinstein (1982) for a justification. A subtlety in applying Rubinstein's model, pointed out by a referee, is that it is not immediately obvious that the threat point should be the value of being a trade in the next period. Effectively we have assumed that if negotiations break down, the traders are forced to wait until the next period to bargain with another agent. See also Rubinstein and Wolinsky (1985).

[4] The reservation value is the value of the item to the seller. Since the use value of a seller is zero, the reservation value is the present value of being a seller without selling this period, which is δV_{t+1}^A.

least one other buyer attending the same auction. In this occasion, competition among buyers occurs. Indeed, a Bertrand game will be played among the buyers. Because of the homogeneity of the buyers, the only solution to the game will be that each buyer gets his reservation value regardless of who wins the bid. Denote the expectation of the buyers and the sellers in the auction market by U_t^A and V_t^A respectively, then the gain from trade in the auction market has an expression of $G_t^A = 1 - \delta U_{t+1}^A - \delta V_{t+1}^A$, where δU_{t+1}^A and δV_{t+1}^A are the reservation values of the buyers and the sellers. The expected utility of a buyer in the auction markets is described by the following equation:

$$\begin{aligned} U_t^A &= e^{-\theta_t(1-x_t)/(1-y_t)}(1 - \delta V_{t+1}^A) + (1 - e^{-\theta_t(1-x_t)/(1-y_t)})\delta U_{t+1}^A \\ &= e^{-\theta_t(1-x_t)/(1-y_t)} G_t^A + \delta U_{t+1}^A. \end{aligned} \tag{3}$$

As an auctioneer, a seller may obtain no buyers, just one buyer, or multiple buyers in a given period. In the first two cases she gets only her reservation value. If there are at least two buyers, competition occurs among buyers and the seller gets $1 - \delta U_{t+1}^A$. Thus, the expected utility of a seller in the auction market must satisfy

$$\begin{aligned} V_t^A &= e^{-\theta_t(1-x_t)/(1-y_t)} \left(1 + \frac{\theta_t(1-x_t)}{1-y_t}\right) \delta V_{t+1}^A \\ &\quad + \left(1 - e^{-\theta_t(1-x_t)/(1-y_t)} \left(1 + \frac{\theta_t(1-x_t)}{1-y_t}\right)\right)(1 - \delta U_{t+1}^A) \\ &= \left(1 - e^{-\theta_t(1-x_t)/(1-y_t)} \left(1 + \frac{\theta_t(1-x_t)}{1-y_t}\right)\right) G_t^A + \delta V_{t+1}^A. \end{aligned} \tag{4}$$

The total value created by trade is the same under either auctions or bargaining when the ratio of buyers and sellers is the same, because the matching technology determines the number of matches and the value of 1 is created every time a seller sells. However, auctions and bargaining distribute the gains from trade differently. Consider a situation where $U_{t+1}^A = U_{t+1}^B$ and $V_{t+1}^A = V_{t+1}^B$, that is, the future value of being a buyer or seller is the same in the two markets, and thus the gain from trade is the same in the markets, $G_{t+1}^A = G_{t+1}^B$. Auctions reward sellers well when two or more buyers appear at the seller relative to bargaining, and poorly when exactly one buyer appears; thus the relative value of auctions over bargaining depends on the relative likelihood of two or more buyers as compared to one buyer. This difference has subtle consequences on any dynamic process governing the choice of institution by buyers and sellers, as we show below.

3. THE EQUILIBRIUM

Beginning with this section, we focus on a situation in which the ratio of stock buyers to sellers is constant over time, i.e., $\theta_t = \theta$ $(t = 0, 1, 2, \ldots)$. Therefore, by market evolution we mean the evolution of the distributions of both the buyers and the sellers in the markets.

We define a steady state market equilibrium as the situation in which the proportions of buyers and sellers in the auction and the bargaining markets are constant all the time, i.e., $(x_t, y_t) = (x, y)$ $(t = 0, 1, 2, \ldots)$. We call such (x, y) a market equilibrium. There are at most three types of market equilibria: (i) $(x, y) = (0, 0)$, i.e., all agents choose auctions; (ii) $(x, y) = (1, 1)$, i.e., all agents select bargaining; and (iii) $(x, y) \in \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$, i.e., a fraction of the buyers and sellers chooses auctions and a fraction of buyers and sellers selects bargaining. It is not difficult to observe that other candidates for equilibrium are impossible. For example, if all sellers are holding auctions while a portion of the buyers are participating in bargaining, then the expected utility for buyers of auctions must be higher than that of bargaining (which in fact is zero). Consequently, all newly arriving buyers would choose auctions rather than bargaining which alters the distribution of buyers in the markets. In other words, all sellers holding auctions while some buyers bargaining cannot be a market equilibrium.

PROPOSITION 1. *For all* θ $(0 < \theta < +\infty)$, $(x, y) = (0, 0)$ *and* $(x, y) = (1, 1)$ *are market equilibria.*

Proofs and derivations are relegated to the Appendix. Proposition 1 confirms that the first two types are actually market equilibria. We explore the possible third type market equilibrium.

At equilibrium, $U_t^i = U^i$ and $V_t^i = V^i$ since $(x_t, y_t) = (x, y)$, hence, $G_t^i = G^i$, where $i = A, B$. Under this circumstance, Eqs. (1)–(4) simplify to:

$$U^B = \frac{1}{2(1-\delta)} \frac{y(1-e^{-\theta x/y})}{\theta x} G^B, \tag{5}$$

$$V^B = \frac{1}{2(1-\delta)} (1-e^{-\theta x/y}) G^B, \tag{6}$$

$$U^A = \frac{1}{1-\delta} e^{-\theta(1-x)/(1-y)} G^A, \tag{7}$$

$$V^A = \frac{1}{1-\delta} \left(1 - e^{-\theta(1-x)/(1-y)} \left(1 + \frac{\theta(1-x)}{1-y}\right)\right) G^A. \tag{8}$$

A necessary condition for a third type equilibrium is that $U^B = U^A$ and $V^B = V^A$. By Eqs. (5)–(8), together with the fact that $G^B = G^A$ (since $U^B = U^A$

and $V^B = V^A$), the condition is equivalent to

$$\frac{y(1 - e^{-\theta x/y})}{2\theta x} = e^{-\theta(1-x)/(1-y)} \tag{9}$$

and

$$\frac{1 - e^{-\theta x/y}}{2} = 1 - e^{-\theta(1-x)/(1-y)} \left(1 + \frac{\theta(1-x)}{1-y}\right), \tag{10}$$

where $0 < x < 1$ and $0 < y < 1$.

PROPOSITION 2. *Let θ_0 be the solution to the equation $e^{\theta} - 1 = 2\theta$. Then, when $\theta \neq \theta_0$, there is no third type equilibrium; when $\theta = \theta_0$, $\{(x, y) \mid 0 < x = y < 1\}$ may also be equilibria.*

Equations (9) and (10) are necessary conditions for (x, y) ($0 < x < 1$ and $0 < y < 1$) to be a market equilibrium. When $\theta \neq \theta_0$, Eqs. (9) and (10) have no solution. This implies that all agents choosing auctions and all agents bargaining are the only two market equilibria. When $\theta = \theta_0$, Eqs. (9) and (10) possess an additional solution set $\{(x, y) \mid 0 < x = y < 1\}$. Note that $\{(x, y) \mid 0 < x = y < 1\}$ are market equilibria provided that when $U^B = U^A$ and $V^B = V^A$ the newly arriving agents choose the markets in such a way that the distributions of buyers and sellers in the markets remain the same. Since new agents are indifferent, this is consistent with optimization. Thus, for all but one value of θ, there are two equilibria. One equilibrium involves all parties bargaining, and it is supported by the absence of trades in the auction market. The other equilibrium involves all parties employing auctions, and it is supported by the absence of trades in the bargaining market.

While both of these equilibria are self-reinforcing, they depend critically on the assumption of unilateral deviations in defining equilibrium. One might reasonably ask whether small multilateral variations will break one of the equilibria, thereby selecting the other. A natural way to pose this question is by imposing an evolutionary dynamic, so that agents respond to utility variations "slowly," and then inquire about the stability of the equilibria under perturbations. It turns out that, under a wide class of evolutionary dynamics, auctions are globally stable.

4. STABILITY OF THE EQUILIBRIA

In what follows we analyze the dynamic properties of the market equilibria. We consider a situation in which the newly arriving agents expect the utilities in the next period to be the same as those of the present period ($U^i_{t+1} = U^i_t$ and $V^i_{t+1} = V^i_t$ $(i = A, B)$), and the ratio of stock buyers to sellers to be stable across time ($\theta_t = \theta$).

These assumptions induce a notion of evolutionary stability. Existing agents remain in the market they chose when new. New agents look at the current matching probabilities and payoffs and choose the market with the highest utility. New agents behave myopically; they assume incorrectly that current payoffs will prevail when in fact the system will evolve. Consequently, the proportion of agents in a market with higher utility in the current period will increase while the proportion of agents in a market with lower utility will decrease. This is a standard evolutionary dynamic (Nachbar, 1990).

In what follows, we ignore the discreteness of the periods in determining the evolutionary dynamic and assume that the size of the entering cohort is small, so that the dynamic is not driven the "step size". As our referees observed, the assumption of small step sizes may be inconsistent with the matching technology, as the matching technology may result in a significant number of exits from the system, and thus steady state requires a significant number of entrants. There are two easy ways to confront this. First, with a discrete number of entrants, most could be required to follow the behavior of departing agents according to their population proportion, with only an insignificant number free to choose the market offering higher rents. A more interesting approach is to limit the number of agents involved in matches, hence reducing the portion of the population that exits. For example, let a fraction σ of the population be "active" in a given period, with inactive agents prohibited from matching. Provided that being active is statistically independent of choices and agent types, the model evolves as before. As the rate of matching is slowed down, agents will discount the future more heavily. In particular, the discount factor δ must not exceed σ provided there is pure time preference.[5]

Under the assumptions, the expected utilities given in (1)–(4) simplify to

$$U_t^B = \frac{1}{2(1-\delta)} \frac{y_t(1-e^{-\theta x_t/y_t})}{\theta x_t} G_t^B, \tag{11}$$

$$V_t^B = \frac{1}{2(1-\delta)} (1-e^{-\theta x_t/y_t}) G_t^B, \tag{12}$$

$$U_t^A = \frac{1}{1-\delta} e^{-\theta(1-x_t)/(1-y_t)} G_t^A, \tag{13}$$

$$V_t^A = \frac{1}{1-\delta} \left(1 - e^{-\theta(1-x_t)/(1-y_t)} \left(1 + \frac{\theta(1-x_t)}{1-y_t}\right)\right) G_t^A, \tag{14}$$

[5] A third alternative involves examining the discrete step size directly. Note that the characterization of utility levels, as represented by Fig. 2, is unchanged. Therefore, provided $\theta \neq \theta_0$, bargaining remains unstable even under discrete step sizes. However, the stability of auctions is now called into question, and in fact, equilibrium cycles may emerge.

which depend solely on the distributions of the buyers and the sellers in current period.

When $U_t^B = U_t^A$, or

$$\frac{y_t(1 - e^{-\theta x_t/y_t})}{\theta x_t} \frac{G_t^B}{2} = e^{-\theta(1-x_t)/(1-y_t)} G_t^A, \tag{15}$$

the buyers are indifferent as to whether they participate in auctions or bargaining. We call Eq. (15) the equilibrium curve for buyers. Similarly, when $V_t^B = V_t^A$, or

$$(1 - e^{-\theta x_t/y_t}) \frac{G_t^B}{2} = \left(1 - e^{-\theta(1-x_t)/(1-y_t)} \left(1 + \frac{\theta(1 - x_t)}{1 - y}\right)\right) G_t^A, \tag{16}$$

the sellers are indifferent between holding auctions and bargaining. We call Eq. (16) the equilibrium curve for sellers. By noting the definitions of G_t^A and G_t^B, Eqs. (15) and (16) can be expressed as (see Appendix for derivation):

$$\frac{(y_t/\theta x_t)(1 - e^{-\theta x_t/y_t})}{2(1 - \delta) + \delta(1 + y_t/\theta x_t)(1 - e^{-\theta x_t/y_t})} = \frac{e^{-\theta(1-x_t)/(1-y_t)}}{1 - \delta e^{-\theta(1-x_t)/(1-y_t)} \theta(1 - x_t)/(1 - y_t)}, \tag{17}$$

$$\frac{1 - e^{-\theta x_t/y_t}}{2(1 - \delta) + \delta(1 + y_t/\theta x_t)(1 - e^{-\theta x_t/y_t})} = \frac{1 - e^{-\theta(1-x_t)/(1-y_t)}(1 + \theta(1 - x_t)/(1 - y_t))}{1 - \delta e^{-\theta(1-x_t)/(1-y_t)} \theta(1 - x_t)/(1 - y_t)}. \tag{18}$$

Note that (x_t, y_t) will evolve over time as a result of trade, termination, and participation of new buyers and sellers. However, because the ratio of the entire stock of buyers to sellers is constant ($\theta_t = \theta$), both the equilibrium curve for buyers and the equilibrium curve for sellers will be fixed along any time path.

The analysis of the equilibrium curves is fundamental to characterizing the evolutionary stability of the equilibria; we perform this analysis using four lemmas.

LEMMA 1. *On both the equilibrium curve for buyers and the equilibrium curve for sellers, x is positively related to y.*

Lemma 1 is not surprising. For instance, suppose sellers are indifferent to participating in auctions or bargaining at (x_0, y_0). An increase in the proportion of buyers in the bargaining market will increase the sellers' likelihood of meeting a buyer in the bargaining market and decrease the probability of having multiple buyers in auctions, therefore increasing the expected utility of sellers in bargaining and decreasing the expected utility of sellers in auctions. To offset the effect of such an increase in x and keep the sellers indifferent between participating in auctions and bargaining, a corresponding increase in y is necessary.

For $0 < \delta < 1$, let θ_1 be the solution to $e^\theta - \delta\theta = 2 - \delta$, and θ_2 be the solution to $e^\theta - 2\theta = (2-\delta)/(1-\delta)$. It is straightforward to show that $\theta_1 < 1 < \theta_0 < \theta_2$.

LEMMA 2. (*a*) *Both the buyers' and sellers' equilibrium curves include* $(1, 1)$ *for all* $0 < \theta < +\infty$;
(*b*) *the buyers' equilibrium curve includes* $(0, y)$, *for some* $y \in (0, 1)$ *when* $\theta < \theta_1$, *and* $(0, 0)$ *when* $\theta \geq \theta_1$;
(*c*) *the sellers' equilibrium curve includes* $(x, 0)$, *for some* $x \in (0, 1)$ *when* $\theta > \theta_2$, *and* $(0, 0)$ *when* $\theta \leq \theta_2$.

Lemma 2 shows that for all $\theta \in (0, +\infty)$, the "upper right end" (see Fig. 1) of both the buyers' and sellers' equilibrium curves is $(1, 1)$. However, the limit of the "lower left end" of the buyers' and sellers' equilibrium curves differ, depending upon the values of δ and θ (note that θ_1 and θ_2 depend on δ). By Lemma 2, the relative position of the lower left ends of the buyers' and the sellers' equilibrium curves can be classified into three categories. (1) For $\theta < \theta_1$, the lower left end of the buyers' equilibrium curve is $(0, y)$ $(0 < y < 1)$ while the lower left end of the sellers' equilibrium curve is $(0, 0)$; (2) for $\theta_1 \leq \theta \leq \theta_2$, the lower left ends of both the buyers' and sellers' equilibrium curves are $(0, 0)$; (3) for $\theta > \theta_2$, the lower left end of the buyers' equilibrium curve is $(0, 0)$ while the lower left end of the sellers' equilibrium curve is $(x, 0)$ $(0 < x < 1)$. When $\theta = \theta_0$ the lower left ends of both the buyers' and sellers' equilibrium curves converge to $(0, 0)$.

An intuition, provided by a referee, arises for Lemma 2 by considering a violation of Lemma 2's claims. Suppose, for example, that the buyer's equilibrium curve included $(1, y)$ for $y < 1$, so that all buyers bargain, but a portion of the sellers go to an auction. This is inconsistent with indifference by the buyers: clearly a buyer going to the auction market would extract all the surplus (facing no competition), while a buyer going to bargaining must split the surplus. The case $(x, 1)$ and the sellers' equilibrium curves are similar.

Now, to see why a point like $(0, y)$ for $y > 0$ might be on the buyers' equilibrium curve, note that $(0, y)$ posits all buyers going to auctions, while some sellers bargain. Suppose that θ is small, so that there are few buyers overall; thus for most of the auctions, there will be only a single buyer who extracts all the surplus. Should a buyer instead go to bargaining, this buyer must split the surplus with a seller, which results in less than full rent extraction by the buyer. Therefore, provided that there are not many buyers overall (technically, this works out to $\theta < \theta_1$), it is possible to equalize the surplus that buyers obtain in auctions with that obtained by buyers going to bargaining, even when the buyers are alone in the auction arena. The case of $(x, 0)$ for $x > 0$ and the sellers' equilibrium curve is analogous.

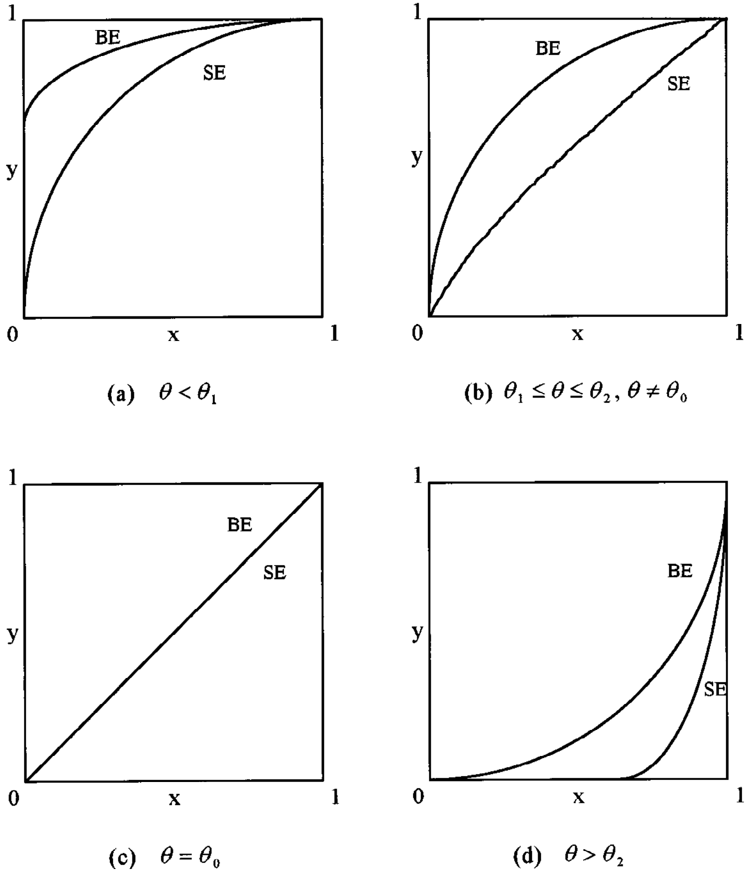

FIGURE 1

LEMMA 3. *Let* $0 < \delta < 1$, *then*

(*a*) *when* $\theta \neq \theta_0$, *the buyers' equilibrium curve always lies above the sellers' equilibrium curve*;

(*b*) *when* $\theta = \theta_0$, *both the buyers' and the sellers' equilibrium curves coincide with the* 45° *line.*

The properties of the buyers' and the sellers' equilibrium curves are illustrated in Fig. 1, where BE represents the buyers equilibrium curve and SE stands for the sellers' equilibrium curve.

Lemma 3(a) is the central theoretical development of this manuscript, in that the major economic results follow from it in a straightforward manner. From

Proposition 2, note that buyers' and sellers' equilibrium curves do not intersect in the interior of the unit square, provided that $\theta \neq \theta_0$. Note as well that these curves are continuous in x, y, and θ. From Lemma 2(b), the buyers' curve lies above the sellers' curve for sufficiently small θ. From Lemma 2(c), the buyers' curve also lies above the sellers' curve for sufficiently large θ. As a consequence of continuity, then, the buyers' curve lies above the sellers' curve for all $\theta \neq \theta_0$.

Moreover, it is intuitive that buyers prefer auctions to bargaining whenever the state (x, y) is below the buyers' equilibrium curve, since this state represents more buyers bargaining than the level which would equalize utility. Similarly, sellers prefer auctions whenever the state lies above the sellers' equilibrium curve. We formalize this in the next result.

LEMMA 4. *(a) $U_t^B < U_t^A$ in the area below the buyers' equilibrium curve, and $U_t^B > U_t^A$ in the area above the curve;*
(b) $V_t^B > V_t^A$ in the area below the sellers' equilibrium curve, and $V_t^B < V_t^A$ in the area above the curve.

The buyers' and sellers' equilibrium curves divide the unit square into three areas. Northwest of the buyers' curve, buyers prefer bargaining and sellers prefer auctions. Between the buyers' curve and the sellers' curve, both types prefer auctions. Finally, southeast of the sellers' curve, buyers prefer auctions and sellers prefer bargaining. The intermediate region is degenerate when $\theta = \theta_0$, and not otherwise.

The existence of the intermediate region is interesting. Consider the case $\theta < \theta_1$, illustrated in Fig. 1a; both the buyers' equilibrium curve and the sellers' equilibrium curve lie above the 45° line (on which the ratio of buyers to sellers is the same). Above the 45° line, there are fewer buyers per seller in bargaining than in auctions. Thus auctions are producing more gains from trade per seller. However, the distribution of this increased surplus is such that both buyers and sellers are better off under auctions.

The expected utilities are indicators of the profitability in the bargaining market and in the auction market. The relative magnitude of the expected utilities dictates the market selection decisions of new agents, and therefore determine the direction of the movement of the stock distributions of buyers and sellers. Northwest of the buyers' equilibrium curve, $U_t^B > U_t^A$ and $V_t^B < V_t^A$, therefore, all newly arriving buyers will choose to bargain while all newly arriving sellers will choose to hold auctions, which increases (decreases) the proportion of buyers in the bargaining (auction) market ($x_{t+1} > x_t$), and increases (decreases) the proportion of sellers in the auction (bargaining) market ($y_{t+1} < y_t$). In the region between two equilibrium curves, $U_t^B < U_t^A$ and $V_t^B < V_t^A$, all new buyers and sellers will choose auctions, which increases (decreases) the proportion of buyers and sellers in the auction (bargaining) market ($x_{t+1} < x_t$, $y_{t+1} < y_t$). Southeast of the sellers' equilibrium curve, $U_t^B < U_t^A$ and $V_t^B > V_t^A$, therefore, all newly arriving buyers will participate in auctions while all newly arriving sell-

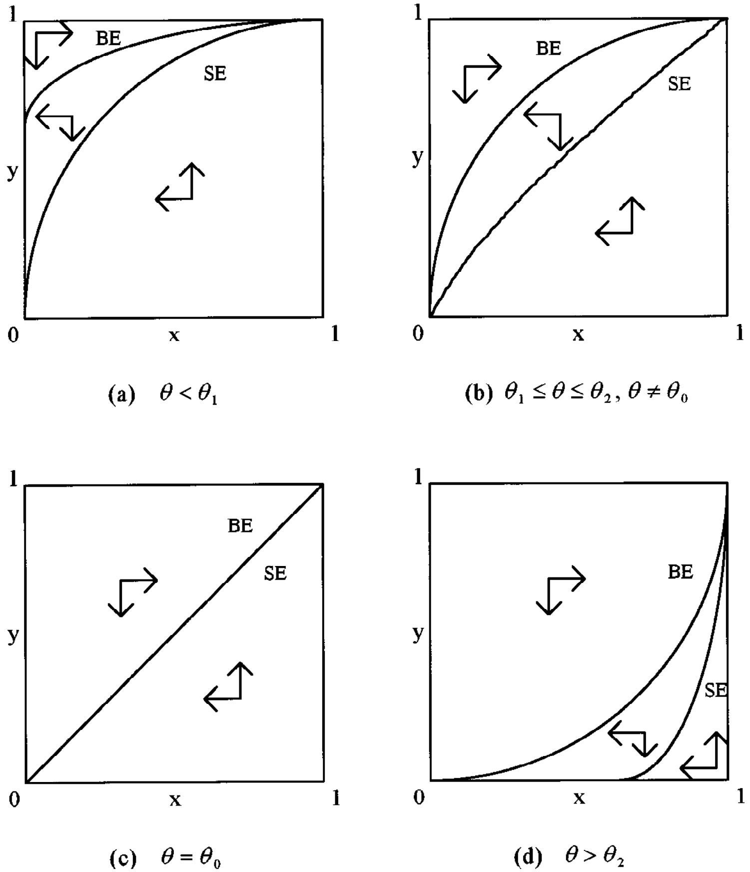

FIGURE 2

ers will choose to bargain, which increases (decreases) the proportion of buyers in the auction (bargaining) market ($x_{t+1} > x_t$), and increases (decreases) the proportion of sellers in the bargaining (auction) market ($y_{t+1} < y_t$). Finally, if it happens that (x_t, y_t) is just on the buyers' equilibrium curve, that is, $U_t^B = U_t^A$ and $V_t^B > V_t^A$, then $x_{t+1} = x_t$ and $y_{t+1} < y_t$; if (x_t, y_t) is just on the sellers' equilibrium curve, that is, $U_t^B < U_t^A$ and $V_t^B = V_t^A$, then $x_{t+1} < x_t$ and $y_{t+1} = y_t$.

The dynamic properties of the distributions of the agents in the markets disclosed by Lemma 4 is displayed in Fig. 2.

Now we turn to analyzing the stability of the equilibria. In Section 3, we have already shown that when $\theta \neq \theta_0$, all agents choosing auctions and all agents

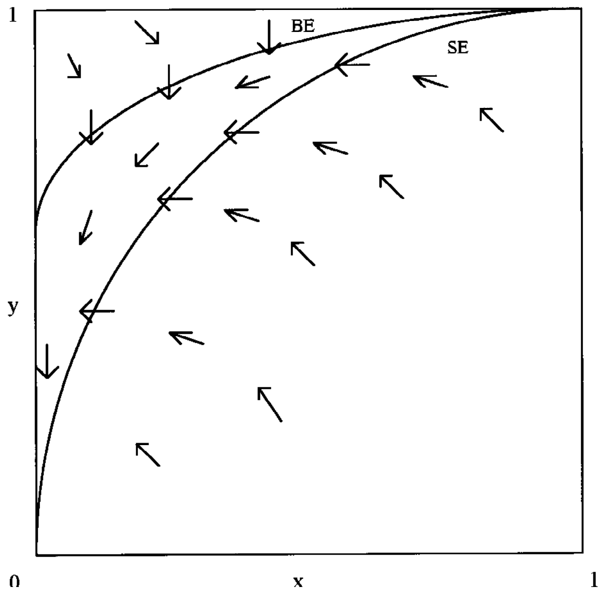

FIGURE 3

selecting bargaining are the only market equilibria; when $\theta = \theta_0$, $\{(x, y) \mid 0 < x = y < 1\}$ are also market equilibria. However, not all of these equilibria are evolutionarily stable.

PROPOSITION 3. (*a*) *When $\theta \neq \theta_0$, all agents choosing auctions is the unique stable equilibrium*;
(*b*) *when $\theta = \theta_0$, there is no locally stable equilibrium.*

Proposition 3 indicates that all sellers holding auctions while all buyers participating in auctions is the unique globally stable equilibrium for all but one θ. In this sense, the auction institution is preferred to the bargaining institution.

The stability of auctions is illustrated in Fig. 3 for the case $\theta < \theta_1$. The dynamic paths cross the buyers' equilibrium curve headed down, with x constant since a buyer's utility is the same in both institutions. Similarly, the dynamic paths cross the sellers' equilibrium curve horizontally. Once between the buyers' and the sellers' equilibrium curves, the path never escapes this region. The other cases are similar.

It is worth noting that the model is robust to the threat of exit, provided that this threat is not too large. For example, suppose that buyers and sellers possess reservation utility levels, and if they fail to achieve these levels, exit with positive probability. Clearly, for low reservation utilities, the stability of auctions and instability of bargaining persists. If there were a distribution of reservation utility levels uncorrelated with other elements of the model, exits would actually enhance the stability of auctions over bargaining in the middle region of Fig. 3, for in this region the bargaining market would suffer more exits for both buyers and sellers. In general, exits merely increase the speed of the system's evolution without changing its direction; therefore stability is not overturned when exits are permitted, provided that the increase in speed is not sufficient to produce discrete steps which admit the possibility of equilibrium cycles.

5. CONCLUSION

In this paper, we develop a model in which homogeneous agents have the ability to choose between the market institutions. While both auctions and bargaining are equilibrium institutions, auctions turn out to be the unique institution stable under a wide class of evolutionary dynamics. This result implies that auctions are superior to bargaining even without the ability of sellers to choose the highest value buyer.

Our results suggest that the auction institution will dominate in the end in a market where factors such as transaction costs of both auctions and bargaining are the same, even if both auction and bargaining institutions are currently prevalent. On the other hand, in markets where bargaining persists, there must be some factors which make bargaining preferable to auctions. The effects of such factors must be strong enough to override the structural advantages of auctions. The major factor in favor of bargaining is the ability of a seller to price discriminate based on observations about the buyers' characteristics. For example, an automobile dealer may offer prices that depend on the buyers' accent; in contrast, auctions are anonymous. Bargaining may also be advantageous when buyers arrive stochastically over time and it is costly to assemble several buyers in the same place, although this does not prohibit the use of sealed-bid auctions, of course. In addition, bargaining may offer the opportunity to alter the terms of the deal; for example, house buyers may bargain over whether a chandelier conveys (that is, comes with the house), while such individualized transactions would be difficult in an auction context.

APPENDIX

Proof of Proposition 1. When all stock agents are playing auctions, all newly arriving agents will participate in auctions instead of bargaining since the expected utility of auctions is higher than that of bargaining. Therefore, once $(x, y) = (0, 0)$ has been reached, there is no tendency to move from it. Similar argument can be applied to $(x, y) = (1, 1)$. ■

Proof of Proposition 2. We need to prove that for all $0 < \theta < +\infty$, $0 < x < 1$, and $0 < y < 1$, Eqs. (9) and (10) have the solution set $\{(x, y, \theta) \mid 0 < x = y < 1, \theta = \theta_0\}$. We first prove that $e^\theta - 1 = 2\theta$ has a unique solution θ_0 in $(0, +\infty)$.

Since $d^2(e^\theta - 1)/d\theta^2 > 0$ and $d^2(2\theta)/d\theta^2 = 0$, the equation has at most two solutions. Zero is obviously one solution so that there is at most one solution in $(0, +\infty)$. On the other hand, since $\lim_{\theta\to+\infty}((e^\theta - 1)/2\theta) = +\infty$ and $e^\theta - 1 < 2\theta$ at $\theta = 1$, it has at least one solution in $(1, \infty)$. In conclusion, the equation has a unique solution in $(1, \infty)$, denoted θ_0.

Let $X = \theta x/y$ and $Y = \theta(1 - x)/(1 - y)$, then Eqs. (9) and (10) become

$$\frac{1 - e^{-X}}{2X} = e^{-Y}, \tag{19}$$

$$\frac{1 - e^{-X}}{2} = 1 - e^{-Y}(1 + Y). \tag{20}$$

We first prove that Eqs. (19) and (20) have a unique solution.

Solving the first equation for Y and then substituting for Y in the second equation we get

$$\frac{1 - e^{-X}}{2} = 1 - \frac{1 - e^{-X}}{2X}\left(1 - \ln\frac{1 - e^{-X}}{2X}\right).$$

Let

$$\begin{aligned} f(X) &= 1 - \frac{1 - e^{-X}}{2X}\left(1 - \ln\frac{1 - e^{-X}}{2X}\right) - \frac{1 - e^{-X}}{2} \\ &= 1 - \frac{1 - e^{-X}}{2X}\left(1 + X - \ln\frac{1 - e^{-X}}{2X}\right), \end{aligned}$$

then

$$\begin{aligned} e^X f(X) &= e^X - \frac{e^X - 1}{2X}\left(1 + X - \ln\frac{1 - e^{-X}}{2X}\right) \\ &= e^X - \frac{e^X - 1}{2X}\left(1 + 2X - \ln\frac{e^X - 1}{2X}\right) \\ &= 1 - \frac{e^X - 1}{2X}\left(1 - \ln\frac{e^X - 1}{2X}\right). \end{aligned}$$

Let $g(Z) = 1 - Z(1 - \ln Z)$, since $dg(Z)/dZ = \ln Z$ and $d^2g(Z)/dZ^2 = 1/Z > 0$, for all $Z > 0$, $g(Z)$ reaches its minimum (zero) at $Z = 1$. Thus, $e^X f(X) \geq 0$, the equality holds only if X satisfies $(e^X - 1)/2X = 1$. We already proved that such X exists, is unique, and is equal to θ_0. Since e^X is positive for all $X > 0$, we conclude that $f(X) \geq 0$ for all $X > 0$. The equality holds only if X equals θ_0.

When $X = \theta_0$, $Y = -\ln((1 - e^{-\theta_0})/2\theta_0) = \theta_0 = X$. Since $X = \theta x/y$ and $Y = \theta(1-x)/(1-y)$, we have $x/y = (1-x)/(1-y)$ or $x = y$, and $\theta = \theta x/y = X = \theta_0$. In other words, Eqs. (9) and (10) have a solution set $\{(x, y, \theta) \mid 0 < x = y < 1, \theta = \theta_0\}$. ■

Derivation of Equations (17) *and* (18). Since

$$G_t^B = 1 - \delta U_t^B - \delta V_t^B = 1 - \frac{\delta}{2(1-\delta)}\left(1 + \frac{y_t}{\theta x_t}\right)(1 - e^{-\theta x_t/y_t})G_t^B,$$

or

$$G_t^B = 1/\left(1 + \frac{\delta}{2(1-\delta)}\left(1 + \frac{y_t}{\theta x_t}\right)(1 - e^{-\theta x_t/y_t})\right),$$

and similarly

$$G_t^A = 1 - \delta U_t^A - \delta V_t^A = 1 - \frac{\delta}{1-\delta}\left(1 - e^{-\theta(1-x_t)/(1-y_t)}\frac{\theta(1-x_t)}{1-y_t}\right)G_t^A,$$

or

$$G_t^A = 1/\left(1 + \frac{\delta}{1-\delta}\left(1 - e^{-\frac{\theta(1-x_t)}{1-y_t}}\frac{\theta(1-x_t)}{1-y_t}\right)\right),$$

the expected utilities can be expressed as

$$U_t^B = \frac{(y_t/\theta x_t)(1 - e^{-\theta x_t/y_t})}{2(1-\delta) + \delta(1 + y_t/\theta x_t)(1 - e^{-\theta x_t/y_t})}, \tag{21}$$

$$V_t^B = \frac{1 - e^{-\theta x_t/y_t}}{2(1-\delta) + \delta(1 + y_t/\theta x_t)(1 - e^{-\theta x_t/y_t})}, \tag{22}$$

$$U_t^A = \frac{e^{-\theta(1-x_t)/(1-y_t)}}{1 - \delta e^{-\theta(1-x_t)/(1-y_t)}\theta(1-x_t)/(1-y_t)}, \tag{23}$$

$$V_t^A = \frac{1 - e^{-\theta(1-x_t)/(1-y_t)}(1 + \theta(1-x_t)/(1-y_t))}{1 - \delta e^{-\theta(1-x_t)/(1-y_t)}\theta(1-x_t)/(1-y_t)}. \tag{24}$$

Therefore, the equilibrium curves for buyers and sellers can be described by Eqs. (17) and (18). ■

Proof of Lemma 1. We need to show that $dy/dx > 0$ for both equilibrium curves. We omit the time index of x and y since both curves are time invariable.

(a) Equation (17) is equivalent to

$$\frac{y}{\theta x}(1-e^{-\theta x/y})\left(e^{-\theta(1-x)/(1-y)}-\delta\frac{\theta(1-x)}{1-y}\right)$$
$$= 2(1-\delta)+\delta\left(1+\frac{y}{\theta x}\right)(1-e^{-\theta x/y}).$$

Totally differentiating both sides of the equation we get

$$\frac{xdy-ydx}{x^2}A+\frac{ydx-xdy}{y^2}B+\frac{-(1-y)dx+(1-x)dy}{(1-y)^2}C=0,$$

where

$$A = \frac{(1-e^{-\theta x/y})}{\theta}\left(e^{\theta(1-x)/(1-y)}-\delta\left(1+\frac{\theta(1-x)}{1-y}\right)\right),$$
$$B = \theta e^{-\theta x/y}\left(\frac{y}{\theta x}\left(e^{\theta(1-x)/(1-y)}-\delta\left(1+\frac{\theta(1-x)}{1-y}\right)\right)-\delta\right),$$
$$C = \frac{y}{x}(1-e^{-\theta x/y})(e^{\theta(1-x)/(1-y)}-\delta).$$

Thus,

$$\frac{dy}{dx}=\left(\frac{yA}{x^2}-\frac{B}{y}+\frac{C}{1-y}\right)\Big/\left(\frac{A}{x}-\frac{Bx}{y^2}+\frac{C(1-x)}{(1-y)^2}\right).$$

Since

$$\frac{yA}{x^2}-\frac{B}{y} = \frac{y}{\theta x^2}\left(\theta A-\frac{\theta x^2 B}{y^2}\right),$$
$$\frac{A}{x}-\frac{Bx}{y^2} = \frac{1}{\theta x}\left(\theta A-\frac{\theta x^2 B}{y^2}\right),$$

while

$$\theta A-\frac{\theta x^2 B}{y^2} = (1-e^{-\theta x/y})\left(e^{\theta(1-x)/(1-y)}-\delta\left(1+\frac{\theta(1-x)}{1-y}\right)\right)$$
$$-\frac{\theta^2 x^2}{y^2}e^{-\theta x/y}\left(\frac{y}{\theta x}\left(e^{\theta(1-x)/(1-y)}-\delta\left(1+\frac{\theta(1-x)}{1-y}\right)\right)-\delta\right)$$
$$= \left(1-e^{-\theta x/y}\left(1+\frac{\theta x}{y}\right)\right)(e^{\theta(1-x)/(1-y)}$$
$$-\delta\left(1+\frac{\theta(1-x)}{1-y}\right)\Big)+\delta\frac{\theta^2 x^2}{y^2}e^{-\theta x/y}>0,$$

hence, $yA/x^2 - B/y > 0$ and $A/x - Bx/y^2 > 0$. On the other hand, $C > 0$, therefore $dy/dx > 0$.

(b) Equation (18) is equivalent to

$$(1 - e^{-\theta x/y}) \left(e^{\theta(1-x)/(1-y)} - \delta \frac{\theta(1-x)}{1-y} \right)$$
$$= \left(2(1-\delta) + \delta \left(1 + \frac{y}{\theta x} \right) \right.$$
$$\left. \times \; (1 - e^{-\theta x/y}) \right) \left(e^{\theta(1-x)/(1-y)} - \left(1 + \frac{\theta(1-x)}{1-y} \right) \right).$$

Totally differentiating both sides of the equation, we get

$$\frac{xdy - ydx}{x^2} A' + \frac{ydx - xdy}{y^2} B' + \frac{-(1-y)dx + (1-x)dy}{(1-y)^2} C' = 0,$$

where

$$A' = -\frac{\delta(1 - e^{-\theta x/y})}{\theta} \left(e^{\theta(1-x)/(1-y)} - \left(1 + \frac{\theta(1-x)}{1-y} \right) \right),$$
$$B' = \theta e^{-\theta x/y} \left(-\frac{\delta y}{\theta x} \left(e^{\theta(1-x)/(1-y)} - \left(1 + \frac{\theta(1-x)}{1-y} \right) \right) \right.$$
$$\left. + \; (e^{\theta(1-x)/(1-y)}(1-\delta) + \delta) \right),$$
$$C' = \theta \left((1 - e^{-\theta x/y})(e^{\theta(1-x)/(1-y)} - \delta) \right.$$
$$\left. - \; \left(2(1-\delta) + \delta \left(1 + \frac{y}{\theta x} \right) (1 - e^{-\theta x/y}) \right) (e^{\theta(1-x)/(1-y)} - 1) \right).$$

Thus,

$$\frac{dy}{dx} = \left(\frac{yA'}{x^2} - \frac{B'}{y} + \frac{C'}{1-y} \right) / \left(\frac{A'}{x} - \frac{B'x}{y^2} + \frac{C'(1-x)}{(1-y)^2} \right).$$

Since

$$\frac{yA'}{x^2} - \frac{B'}{y} = \frac{y}{\theta x^2} \left(\theta A' - \frac{\theta x^2 B'}{y^2} \right),$$
$$\frac{A'}{x} - \frac{B'x}{y^2} = \frac{1}{\theta x} \left(\theta A' - \frac{\theta x^2 B'}{y^2} \right),$$

while

$$\theta A' - \frac{\theta x^2 B'}{y^2} = -\delta(1 - e^{-\theta x/y}) \left(e^{\theta(1-x)/(1-y)} - \left(1 + \frac{\theta(1-x)}{1-y} \right) \right)$$

$$-\frac{\theta^2x^2}{y^2}e^{-\theta x/y}\left(-\frac{\delta y}{\theta x}\left(e^{\theta(1-x)/(1-y)}-\left(1+\frac{\theta(1-x)}{1-y}\right)\right)\right.$$
$$\left.+\ (e^{\theta(1-x)/(1-y)}(1-\delta)+\delta)\right)$$
$$= -\delta\left(1-e^{-\theta x/y}\left(1+\frac{\theta x}{y}\right)\right)$$
$$\times\left(e^{\theta(1-x)/(1-y)}-\left(1+\frac{\theta(1-x)}{1-y}\right)\right)$$
$$-\frac{\theta^2x^2}{y^2}e^{-\theta x/y}(e^{\theta(1-x)/(1-y)}(1-\delta)+\delta) < 0,$$

hence $yA'/x^2 - B'/y < 0$ and $A'/x - B'x/y^2 < 0$.

On the other hand, let $X - \theta(1-x)/(1-y)$; by using Eq. (18) we obtain

$$C' = \theta\left((1-e^{-\theta x/y})(e^{\theta(1-x)/(1-y)}-\delta)\right.$$
$$\left.-\left(2(1-\delta)+\delta\left(1+\frac{y}{\theta x}\right)(1-e^{-\theta x/y})\right)(e^{\theta(1-x)/(1-y)}-1)\right)$$
$$= K((e^X-\delta)(e^X-(1+X))-(e^X-1)(e^X-\delta X))$$
$$= K(-Xe^X(1-\delta)-\delta(e^X-(1+X))-\delta X) < 0,$$

where, $K = 2(1-\delta)+\delta(1+\frac{y}{\theta x})(1-e^{-\theta x/y})/(e^X-\delta X)$, therefore $dy/dx > 0$. ■

Proof of Lemma 2. (a) Suppose y does not go to 1 when $x \to 1$, but $y \to y'$, where $0 \le y' < 1$. By Lemma 1, $y' \neq 0$.

When $(x, y) \to (1, y')$, Eq. (17) becomes

$$\frac{(y'/\theta)(1-e^{-\theta/y'})}{2(1-\delta)+\delta(1+y'/\theta)(1-e^{-\theta/y'})} = 1,$$

or

$$\frac{y'}{\theta}(1-e^{-\theta/y'}) = 2+\frac{\delta}{1-\delta}(1-e^{-\theta/y'}).$$

But this equation does not hold since the left-hand side of the equation cannot exceed 1 while the right-hand side is at least 2, no matter which values y and θ take.

When $(x, y) \to (1, y')$, Eq. (18) becomes

$$\frac{1-e^{-\theta/y'}}{2(1-\delta)+\delta(1+y'/\theta)(1-e^{-\theta/y'})} = 0.$$

But this equation obviously does not hold since the right-hand side of the equation is larger than zero, no matter which values y and θ take. Therefore, for both equilibrium curves, it is true that $(x, y) \to (1, 1)$ for all $\delta \in (0, 1)$ and $\theta \in (0, \infty)$. In other words, the upper right ends of both buyers' and sellers' equilibrium curves converge to $(1, 1)$.

(b) When $(x, y) \to (0, y')$, where $0 < y' < 1$, Eq. (17) becomes $1/(2 - \delta) = e^{-w}/(1 - \delta e^{-w} w)$ or $2 - \delta = e^w - \delta w$, where $w = \theta/(1 - y')$. Since $(d/dw)(e^w - \delta w) > 0$, $(e^w - \delta w)|_{w=0} = 1 < 2 - \delta$ and $(e^w - \delta w)|_{w=\infty} = \infty$, it has a unique solution $w = \theta_1$. Thus, $\theta/(1 - y') = \theta_1$ or $y' = 1 - \theta/\theta_1$. Therefore, when $\theta < \theta_1$, the equation has a unique solution $y' \in (0, 1)$. In other words, when $\theta < \theta_1$, the lower left end of the buyers' equilibrium curve converges to $(0, y')$.

The lower left end of the buyers' equilibrium curve does not converge to $(x', 0)$, where $x' \in (0, 1)$ since Eq. (17) does not hold at $(x, y) = (x', 0)$ for all $\theta \in (0, \infty)$. Therefore, when $\theta \geq \theta_1$, the lower left end of the buyers' equilibrium curve converges to $(0, 0)$.

(c) When $(x, y) \to (x', 0)$, where $0 < x' < 1$, Eq. (18) becomes $1/(2-\delta) = (1 - e^{-z}(1 + z))/(1 - \delta e^{-z} z)$ or $e^z = (2 - \delta)/(1 - \delta) + 2z$, where $z = \theta(1 - x')$. Since $d^2 e^z/dz^2 = e^z > 0$ and $(d^2/dz^2)((2 - \delta)/(1 - \delta) + 2z) = 0$, while $e^z|_{z=-\infty} > ((2 - \delta)/(1 - \delta) + 2z)|_{z=-\infty}$, $e^z|_{z=0} < ((2 - \delta)/(1 - \delta) + 2z)|_{z=0}$, and $e^z|_{z=\infty} > ((2 - \delta)/(1 - \delta) + 2z)|_{z=\infty}$, it has a unique solution $z = \theta_2 \in (0, \infty)$. Thus, $\theta(1 - x') = \theta_2$ or $x' = 1 - \theta_2/\theta$. Therefore, when $\theta > \theta_2$, the equation has a unique solution $x' \in (0, 1)$. In other words, when $\theta > \theta_2$, the lower left end of the buyers' equilibrium curve converges to $(x', 0)$.

The lower left end of the buyers' equilibrium curves does not converge to $(0, y')$, where $y' \in (0, 1)$ since Eq. (18) does not hold at $(x, y) = (0, y')$ for all $\theta \in (0, \infty)$. Therefore, when $\theta \leq \theta_2$, the left end of the buyers' equilibrium curve converges to $(0, 0)$. ■

Proof of Lemma 3. By Lemma 2, the claim of Lemma 3 is true when $\theta < \theta_1$ or $e^\theta - \delta\theta < 2 - \delta$ because the lower left end of the buyers' curve is above the sellers' curve. We consider $\theta \geq \theta_1$ or $e^\theta - \delta\theta \geq 2 - \delta$.

Let $y = \frac{1}{2}$ and the corresponding points on the buyers' and the sellers' equilibrium curves be x_1 and x_2 respectively. By Eqs. (17) and (18), x_1 and x_2 are the solution to the following equations.

$$\frac{1 - e^{-2\theta x_1}}{2(1 - \delta) + \delta(1 + 1/2\theta x_1)(1 - e^{-2\theta x_1})} = \frac{2\theta x_1}{e^{2\theta(1 - x_1)} - 2\delta\theta(1 - x_1)},$$

$$\frac{1 - e^{-2\theta x_2}}{2(1 - \delta) + \delta(1 + 1/2\theta x_2)(1 - e^{-2\theta x_2})} = \frac{e^{2\theta(1 - x_2)} - (1 + 2\theta(1 - x_2))}{e^{2\theta(1 - x_2)} - 2\delta\theta(1 - x_2)}.$$

According to Lemma 1, such x_1 and x_2 are unique. We need to prove that $x_1 \leq x_2$.

Let

$$f(x) = \frac{1-e^{-2\theta x}}{2(1-\delta)+\delta(1+1/2\theta x)(1-e^{-2\theta x})},$$
$$g(x) = \frac{2\theta x}{e^{2\theta(1-x)}-2\delta\theta(1-x)},$$

and

$$h(x) = \frac{e^{2\theta(1-x)}-(1+2\theta(1-x))}{e^{2\theta(1-x)}-2\delta\theta(1-x)}.$$

Then,

$$\begin{aligned}
\frac{df(x)}{dx} &= \frac{1}{(2(1-\delta)+\delta(1+1/2\theta x)(1-e^{-2\theta x}))^2} \\
&\quad \times \left(2\theta e^{-2\theta x}\left(2(1-\delta)+\delta\left(1+\frac{1}{2\theta x}\right)(1-e^{-2\theta x})\right)\right. \\
&\quad \left. -\delta(1-e^{-2\theta x})\left(-\frac{1}{2\theta x^2}(1-e^{-2\theta x})+\left(1+\frac{1}{2\theta x}\right)2\theta e^{-2\theta x}\right)\right) \\
&= \frac{4(1-\delta)\theta e^{-2\theta x}+\delta(1-e^{-2\theta x})^2/2\theta x^2}{(2(1-\delta)+\delta(1+1/2\theta x)(1-e^{-2\theta x}))^2} > 0. \\
\frac{dg(x)}{dx} &= \frac{2\theta(e^{2\theta(1-x)}-2\delta\theta(1-x)-x(-2\theta e^{2\theta(1-x)}+2\delta\theta))}{(e^{2\theta(1-x)}-2\delta\theta(1-x))^2} \\
&= \frac{2\theta}{(e^{2\theta(1-x)}-2\delta\theta(1-x))^2}(e^{2\theta(1-x)}(1+2\theta x)-2\delta\theta),
\end{aligned}$$

but

$$\frac{d}{dx}(e^{2\theta(1-x)}(1+2\theta x)-2\delta\theta) = e^{2\theta(1-x)}(-4\theta^2 x) < 0,$$

while

$$(e^{2\theta(1-x)}(1+2\theta x)-2\delta\theta)\Big|_{x=1} = 1+2\theta(1-\delta) > 0.$$

Thus,

$$\frac{dg(x)}{dx} > 0.$$

In other words, both $f(x)$ and $g(x)$ are strictly increasing.

At $x = 1$,

$$f(1) = \frac{1 - e^{-2\theta}}{2(1-\delta) + \delta(1 + 1/2\theta)(1 - e^{-2\theta})},$$
$$g(1) = 2\theta,$$

hence

$$\frac{g(1)}{f(1)} = 2\theta \frac{2(1-\delta) + \delta(1 + 1/2\theta)(1 - e^{-2\theta})}{1 - e^{-2\theta}}$$
$$= \frac{4\theta(1-\delta)}{1 - e^{-2\theta}} + \delta(1 + 2\theta).$$

Since

$$\frac{d}{d\theta}\left(\frac{4\theta(1-\delta)}{1 - e^{-2\theta}} + \delta(1 + 2\theta)\right) = \frac{4(1-\delta)}{(1 - e^{-2\theta})^2}\left(1 - \frac{1 + 2\theta}{1 - e^{-2\theta}}\right) + 2\delta > 0$$

so that

$$\frac{g(1)}{f(1)} > \left(\frac{4\theta(1-\delta)}{1 - e^{-2\theta}} + \delta(1 + 2\theta)\right)\bigg|_{\theta=0} = 2 - \delta > 1,$$

therefore $g(1) > f(1)$.

At $x = 0$, it is easy to see that $f(0) = g(0)$. In addition, since

$$\frac{df(x)}{dx} = \frac{4(1-\delta)\theta e^{-2\theta x} + \delta(1 - e^{-2\theta x})^2/2\theta x^2}{(2(1-\delta) + \delta(1 + 1/2\theta x)(1 - e^{-2\theta x}))^2}\bigg|_{x=0} = \frac{2\theta}{2 - \delta}$$

and

$$\frac{dg(x)}{dx} = \frac{2\theta(e^{2\theta(1-x)}(1 + 2\theta x) - 2\delta\theta)}{(e^{2\theta(1-x)} - 2\delta\theta(1-x))^2}\bigg|_{x=0} = \frac{2\theta}{e^{2\theta} - 2\delta\theta},$$

when $\theta \geq \theta_1$, $e^{2\theta} - 2\delta\theta > e^{\theta} - \delta\theta \geq 2 - \delta$, we have $df(0)/dx > dg(0)/dx$. Therefore, $f(x) > g(x)$ for $x \in (0, x_1)$ and $f(x) < g(x)$ for $x \in (x_1, 1)$.

On the other hand,

$$\frac{dh(x)}{dx} = \frac{1}{(e^{2\theta(1-x)} - 2\delta\theta(1-x))^2}((-2\theta e^{2\theta(1-x)} + 2\theta)(e^{2\theta(1-x)} - 2\delta\theta(1-x))$$
$$-(e^{2\theta(1-x)} - (1 + 2\theta(1-x)))(-2\theta e^{2\theta(1-x)} + 2\delta\theta))$$
$$= \frac{4\theta^2(\delta - 1)(1-x)e^{2\theta(1-x)} + 2\delta\theta(1 - e^{2\theta(1-x)})}{(e^{2\theta(1-x)} - 2\delta\theta(1-x))^2} < 0,$$

that is, $h(x)$ is strictly decreasing. Meanwhile, $h(1) = 0$ and $h(0) = (e^{2\theta} - (1 + 2\theta))/(e^{2\theta} - 2\delta\theta) > 0$.

By the properties of $f(x)$, $g(x)$, and $h(x)$, we only need to prove that $f(x^*) \leq g(x^*)$, where x^* is the unique solution to the equation $g(x) = h(x)$. Solving the equation $g(x) = h(x)$ we get $x^* = 1 - \ln(1 + 2\theta)/2\theta$. Since

$$
\begin{aligned}
f(x) - g(x) &= \frac{1 - e^{-2\theta x}}{2(1-\delta) + \delta(1 + 1/2\theta x)(1 - e^{-2\theta x})} \\
&\quad - \frac{2\theta x}{e^{2\theta(1-x)} - 2\delta\theta(1-x)} \\
&= \frac{(1-e^{-2\theta x})(e^{2\theta(1-x)} - 2\delta\theta(1-x) - \delta(1+2\theta x)) - 4\theta(1-\delta)x}{K_1 K_2},
\end{aligned}
$$

where $K_1 = 2(1-\delta) + \delta(1 + 1/2\theta x)(1 - e^{-2\theta x})$ and $K_2 = e^{2\theta(1-x)} - 2\delta\theta(1-x)$,

$$
\begin{aligned}
f(x^*) - g(x^*) &= \frac{1}{K_1 K_2}((1 - e^{-2\theta}(1 + 2\theta))((1 + 2\theta) - \delta\ln(1 + 2\theta) \\
&\quad -\delta(1 + 2\theta - \ln(1 + 2\theta))) - 2(1-\delta)(2\theta - \ln(1 + 2\theta))) \\
&= \frac{1}{K_1 K_2}((1 - e^{-2\theta}(1 + 2\theta))(1-\delta)(1 + 2\theta) \\
&\quad -2(1-\delta)(2\theta - \ln(1 + 2\theta))) \\
&= \frac{1-\delta}{K_1 K_2}(1 - e^{-2\theta}(1 + 2\theta)^2 - 2\theta + 2\ln(1 + 2\theta)) \\
&= \frac{1-\delta}{K_1 K_2}\left(1 - \left(\frac{1+2\theta}{e^{\theta}}\right)^2 + \ln\left(\frac{1+2\theta}{e^{\theta}}\right)^2\right).
\end{aligned}
$$

Let $F(X) = 1 - X + \ln X$, then $dF(X)/dX = -1 + 1/X$, $d^2F(X)/dX^2 = -1/X^2 < 0$. Thus, $F(X)$ is maximized at $X = 1$ with the value of $F(1) = 0$. Because $K_1 > 0$ and $K_2 > 0$, when $\theta \neq \theta_0$, $(1 + 2\theta)/e^{\theta} \neq 1$, therefore $f(x^*) - g(x^*) < 0$, hence $x_1 < x_2$. When $\theta = \theta_0$, $(1 + 2\theta)/e^{\theta} = 1$, therefore $f(x^*) - g(x^*) = 0$, hence $x_1 = x_2$. Indeed, by Proposition 2, when $\theta = \theta_0$, the buyers' equilibrium curve coincides with the sellers' equilibrium curve at the 45° line. ■

Proof of Lemma 4. By Eqs. (11) and (13),

$$
\lim_{\substack{x_t \to 1 \\ y_t \to 0}} U_t^B = 0 < 1 = \lim_{\substack{x_t \to 1 \\ y_t \to 0}} U_t^A.
$$

Since U_t^B and U_t^A are continuous, we conclude that $U_t^B < U_t^A$ in the area below the buyers' equilibrium curve, and $U_t^B > U_t^A$ in the area above the curve. On the other hand, by Eqs. (12) and (14), $\lim_{\substack{x_t \to 1 \\ y_t \to 0}} V_t^B = 1/(2-\delta) > 0 = \lim_{\substack{x_t \to 1 \\ y_t \to 0}} V_t^A$. Since V_t^B and V_t^A are continuous, we conclude that $V_t^B > V_t^A$ in the area below the sellers' curve, and $V_t^B < V_t^A$ in the area above the curve. ■

Proof of Proposition 3. (a) $\theta \neq \theta_0$. Suppose (x_t, y_t) is the state of distributions of buyers and sellers in period t, where $0 < x_t < 1$ and $0 < y_t < 1$. If (x_t, y_t) is on the buyers' equilibrium curve, then $x_{t+1} = x_t$ and $y_{t+1} < y_t$. If (x_t, y_t) is on the sellers' equilibrium curve, then, $x_{t+1} < x_t$ and $y_{t+1} = y_t$. In both cases, (x_{t+1}, y_{t+1}) will be in between the buyers' and the sellers' equilibrium curves. If (x_t, y_t) is in between the buyers' and sellers' equilibrium curves, then, $x_{t+1} < x_t$ and $y_{t+1} < y_t$. Hence, (x_{t+1}, y_{t+1}) will remain in between the buyers' and sellers' equilibrium curves. Therefore, once (x_t, y_t) falls between or on either one of the curves, it will never leave the region again. Notice that x_t and y_t are strictly decreasing in this region. Therefore, (x_t, y_t) converges to $(0, 0)$. On the other hand, if (x_t, y_t) is above the buyers' equilibrium curve, then $x_{t+1} > x_t$ and $y_{t+1} < y_t$. Such movement will continue until (x_t, y_t) happens to hit the buyers' equilibrium curve or falls between the buyers' and sellers' equilibrium curves. By the above argument, eventually it also converges to $(0, 0)$. A similar argument applies to the case when (x_t, y_t) is below the sellers' equilibrium curve. In conclusion, for all (x_t, y_t) $(0 < x_t < 1$ and $0 < y_t < 1)$, (x_t, y_t) will converge to $(0, 0)$.

(b) $\theta = \theta_0$. In this case, the buyers' equilibrium curve coincides with the sellers' equilibrium curve. If (x_t, y_t) is above or below the buyers' equilibrium curve (the sellers' equilibrium curve), (x_t, y_t) will converge to an equilibrium point on the curve, which is not necessarily the equilibrium point it originally deviated from (it is a zero probability event). Therefore, no equilibrium is stable when $\theta = \theta_0$. ■

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