Mechanism Design by Competing Sellers

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A dynamic model with many sellers and many buyers is constructed, in which buyers who fail to purchase in the current period may attempt to purchase in the future, and sellers who fail to sell may sell in the future. An equilibrium is found where sellers hold identical auctions and buyers randomize over the sellers they visit. Auctions alter the distribution of buyer types by removing high value buyers more rapidly than low value buyers, and an equilibrium distribution of buyer types is constructed. Sellers in equilibrium post an efficient reserve price equal to the sellers' value of the good, and an auction with efficient reserve is an optimal mechanism from each seller's point of view, in spite of the ability of any seller to alter the distribution of buyer types participating in the seller's mechanism by altering the mechanism.

KEYWORDS: Auction, competitive equilibrium, asymmetric information, mechanism design, price formation.

1. INTRODUCTION

THE STANDARD AUCTION MODEL[2] posits a monopoly seller of a single good. In contrast, this paper considers a model in which many sellers, each with a single good to sell, compete for buyers. At any given time, each buyer may participate in at most one seller's mechanism. Thus, unlike the standard model, a seller must offer surplus to the buyer sufficient to attract the buyer away from alternative sellers, and this surplus is determined endogenously, by the equilibrium auctions or mechanisms employed by other sellers.

The literature on price formation divides naturally into two categories. In the first, the seller of a unique item designs a mechanism to maximize his expected profits. In the second category, many sellers compete according to a fixed set of rules.[3] The present study differs from the first category by introducing competing sellers, but retains the endogeneity of the selling institution by allowing sellers to choose mechanisms as part of the game. The present study differs from the second category in three significant respects. With the exception of the Bertrand model, extant models of competing sellers impose an exogenous technology matching buyers and sellers. In contrast, I will place sellers at fixed locations and allow buyers to choose which seller they go to in each period. In

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[2] See McAfee and McMillan (1987) for a survey.

[3] The Bertrand model is the first model to incorporate agents who choose prices. More recent treatments, for example, Diamond (1982), Rubinstein and Wolinsky (1985), Gale (1986), and Wolinsky (1988), introduce matching and market dynamics, but prohibit buyers from choosing their preferred seller, in favor of an exogenous matching technology. The present model may be viewed as a capacity constrained Bertrand model with an enriched strategy space for the sellers.

equilibrium, all sellers will use identical auctions, and buyers will randomize over the set of sellers. However, the ability of a seller to deviate and employ a different mechanism, for example, by lowering the reserve price, imposes an equilibrium condition which would not be present were an exogenous matching technology imposed. In this model, this equilibrium condition forces sellers to post an efficient reserve price, so that the reserve price equals the value of the item to the seller, which in turn equals the present discounted value of being a seller in the next period.

The second significant difference of the present study involves the endogeneity of the transaction mechanism. In the existing literature, the mechanism is typically Nash bargaining, although more recent studies, for example Rubinstein and Wolinsky (1985), allow for strategic behavior within a given bargaining framework (alternating offers). In the present model, sellers may choose any mechanism for transaction. Thus, the transaction institution, in this case, an auction, arises endogenously, as in the first category of the price formation literature.

The final difference is that the model allows for one-sided asymmetric information. That is, buyers know their own valuation for the seller's good, and this differs across buyers. Because of the revenue equivalence theorem,[4] the type of auction will not be uniquely determined. Auctions tend to remove high value buyers more rapidly than low value buyers; the equilibrium value distribution of the stock of buyers will not generally coincide with the value distribution of an entering cohort of buyers. Thus, allowing buyer valuations to differ across buyers permits the value distribution in the pool of buyers to be endogenously determined.[5]

The analysis concentrates on a steady state environment. The analytic problems described by Gale (1986) for this kind of model basically do not arise, with one exception. If all sellers hold auctions with constant reserve price in every period, then the steady state is globally stable. However, away from the steady state, the seller's desire to hold an auction persists, but the equilibrium reserve price may vary as the system evolves, and it does not appear feasible to prove global stability as the reserve price varies. This weakens, but does not vitiate, the validity of analyzing steady states.

The structure of the paper is as follows. The second section presents the model, the equilibrium concept, and some mathematical preliminaries. The third section identifies conditions when auctions comprise a unique symmetric equilibrium in any period. The fourth section considers large economy steady

[4] The revenue equivalence theorem states that, with symmetric independently distributed private buyer values, all the common auction forms produce the same expected rents for the seller and buyers, and, with appropriate reserve price, maximize the seller's expected profits, given a condition on the distribution of buyer types. See Milgrom and Weber (1982) and Myerson (1981).

[5] Wolinsky (1988) allows for one-sided asymmetric information. His model differs from the present study by the first two differences described above. In addition, Wolinsky's buyers receive new draws in each period, so the value distribution in the stock of buyers is not endogenous.

state equilibrium. Comparative statics are offered in the fifth section, and the model's defects are discussed in the conclusion.

2. PRELIMINARIES

I begin with an informal description of the model, with the exact meaning of some of the terms clarified below. Periods in the model are nonnegative integers $t = 0, 1, 2, \ldots$. At the beginning of a period, there is a stock of buyers and sellers inherited from the previous period. The first action of each period is for sellers to simulataneously choose mechanisms from the class of direct mechanisms (see Myerson (1982)). These mechanisms map buyers' reported types into feasible outcomes, that is, who gets the good and how much each agent pays. The mechanisms are constrained to be *anonymous* with respect to the buyers, that is, they cannot distinguish among different buyers except on the basis of the reports by buyers to the mechanism.[6] Each mechanism must specify the outcome as a function of both the buyers' reported types and the number of buyers, because at the time of selecting the mechanism, the seller does not know how many buyers will be participating in the mechanism. The mechanisms are constrained by the fact that each seller has only one unit of the good to sell. Sellers have zero use value for their unit of the good.

Once sellers have chosen their mechanisms, these mechanisms are announced to the buyers. Buyers simultaneously choose which mechanism they will participate in, and they may choose at most one in a given period. Buyers then arrive at the mechanism. Each mechanism takes reports from the buyers, and dictates the allocation of the good and payments made by the buyers. Buyers are assumed to not know the realization of the number of buyers participating in the mechanism they chose to participate in, although of course in equilibrium they know the distribution of the number of participants. The mechanisms are operated on the set of participating buyers. If the mechanism dictates that a buyer receive the good, both the buyer that receives the good and the seller exit from the game. Other buyers remain in the stock of buyers. If the mechanism dictates that the seller retain the good, all buyers and the seller remain in the stock of buyers and sellers, respectively. Once the mechanisms have operated, nature operates on the existing stocks, removing each buyer and seller with probability $(1 - \alpha)$ independently of the history of the game. Once nature has moved, S new sellers and bS new buyers are added to the stock of buyers and sellers, respectively. The entry of new buyers and sellers ends the period, and the next period begins with the current stock of sellers choosing mechanisms.

Each new seller has one unit of the good, and each new buyer has a valuation for one unit of the good which is drawn from the cumulative distribution

[6] This assumption amounts to assuming that the seller does not know the buyers' names, nor does the seller have any way to identify buyers except by the buyers' own reports to the mechanism. It is a useful assumption because it prohibits the seller from choosing a mechanism which does not admit a symmetric equilibrium among the buyers; that is, all buyers of a given type use the same strategy.

function F, and each entering buyers' valuation is drawn independently of the history of the game and of the other new buyers' valuations. Once endowed with a valuation, a buyer's valuation does not change. The distribution function F is assumed to have a continuous density f and f has support $[0,1]$. Buyers and sellers have linear utility and apply discount δ^t to period t.

The analysis combines elements of game theoretic analysis and of traditional competitive-equilibrium analysis. Along the lines of traditional competitive-equilibrium analysis, it is assumed that agents neglect certain strategic repercussions of their actions because they are small when the economy is large. Specifically, all agents neglect the effects of the actions of any one seller on the surplus available to buyers who do not participate in the seller's mechanism. In addition, all agents neglect the effect of the behavior of a single seller on available future profits. In contrast, strategic interdependencies within a group of buyers participating in a given mechanism are fully taken into account. Analysis of these interdependencies is simplified by a condition related to Harsanyi and Selten's (1988) notion of *subgame consistency*. Subgame consistency, when combined with a single-valued solution concept, requires that identical subgames will have the same solution. Since the present model involves private information, the appropriate notion here is subform consistency, which requires that all identical subforms are played in the same way. In the present model, this will produce stationary strategies.

Specifically, suppose that seller i deviates in period t from the equilibrium path, by offering some other mechanism. I assume

(i) that all agents believe the expected profits in period t of buyers who do not participate in seller i's mechanism to be invariant to seller i's choice of mechanism, and

(ii) that all agents believe the expected profits associated with future periods are invariant to the deviation of a seller in the current period.

Given the beliefs implied by (i) and (ii), and the expected behavior of other agents, each agent maximizes expected utility, and I call such behavior a *Competitive Subform Consistent Equilibrium* (CSCE).

In a large economy, it is clear that any one seller's choice of mechanism on the utility of buyers who don't participate in that mechanism will be small, of order $1/m$, where m is the number of sellers. However, in the formulation of equilibrium, I will force the sellers to neglect this effect entirely, which is analogous to the presumption in competitive equilibrium that all agents neglect their effect on prices. The analogy is quite close, because the utility a buyer obtains at an alternative seller is the "price" that a seller has to pay to attract the buyer to his mechanism, as the seller must offer the buyer at least as much utility as that buyer obtains at an alternative mechanism. Assumption (i) should be satisfied in an economy with infinitely many sellers. This assumption is not otherwise defensible, and is made only for tractability.

Assumption (ii) permits the continuation values in the model to be embedded in the current period by way of functions invariant to the behavior in the current period. This implies several restrictions. First, a deviating seller believes he can return to the equilibrium path, if he doesn't sell in the current period. Second, to attract a buyer in the current period, a seller must offer more than the discounted value of being a buyer in a future period. This amount is assumed invariant to what the seller actually offers. The assumption implies a strong form of subform consistency, in that the option value of the future is not presumed to change when a deviation occurs. Assumption (ii) is also unreasonable in an economy with finitely many players, but is reasonable with infinitely many players, and in this circumstance it corresponds to stationarity of beliefs in the face of deviations.

REMARK 1: Assumption (ii) guarantees that, in any given period t, there is a value Φ^* of being a seller in the next period, and a function π^*_{t+1} which gives the value of being a buyer in the next period as a function of the buyer's valuation, and that these are invariant to behavior period t.

I will restrict sellers to use direct anonymous mechanisms, that is, their mechanisms cannot depend on the identity of the buyer. As a result, the functions used in seller mechanisms, such as the price or probability of obtaining the object, must satisfy an invariance property with respect to permutations of the buyers' reports. For $x \in \mathbb{R}^n$, let $(x_j, x_i, x_{-i,j})$ represent the permutation vector $(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n)$. If $A \subseteq \mathbb{R}^n$, a function f: $A \to \mathbb{R}^n$ is *permutation invariant* if

$$f(x) = \left(f_j(x_j, x_i, x_{-i,j}), f_i(x_j, x_i, x_{-i,j}), f_{-i,j}(x_j, x_i, x_{-i,j})\right),$$

where f_k are the component functions of f. That is, f is permutation invariant if permuting x permutes $f(x)$ in the same fashion.

If n buyers show up at a given seller's mechanism, the seller is permitted to ask the buyers for reports of their values, and then can charge the ith buyer an amount P_i, and offer a probability of obtaining the good of Q_i, which may depend on the entire vector of reports. A direct mechanism must specify these functions, for every integer n, up to the maximum possible number n_t. A *direct anonymous mechanism* is a set of functions $\{(Q^n, P^n)\}_{n=1}^{n_t}$, where Q^n: $[0,1]^n \to [0,1]^n$ and P^n: $[0,1]^n \to \mathbb{R}^n$ are permutation invariant and Q^n satisfies

$$(1) \qquad \left(\forall x \in [0,1]^n\right) \quad \sum_{i=1}^{n} Q_i^n(x) \leqslant 1.$$

In the event that n potential buyers participate in the seller's mechanism $\{(Q^n, P^n)\}_{n=1}^{n_t}, Q^n$ gives the vector of probabilities that the participating agents receive the item, and (1) dictates that these probabilities must sum to no more than unity because the seller has one unit. These probabilities are functions of the vector of reports. An *action* for a seller in period t is a direct anonymous mechanism.[7] A *strategy* for a seller is a mapping from histories into the set of probability distributions over direct anonymous mechanisms in each period of the game. History will not play much of a role in the analysis because in a subform consistent equilibrium, history affects agents' behaviors only as it affects the data of the subform that remains to be played. The only relevant data on the subforms are the number of sellers and the distribution of buyer types, which are not affected by deviations by a single seller.

If a seller receives the payments p_t in period t and then sells the good or is removed in period T, the seller's utility is $\sum_{t=0}^{T}\delta^t p_t$. If a buyer makes payments y_t in period t and then purchases in period T, and has value x, the buyer's utility is $\delta^T x - \sum_{t=0}^{T}\delta^t y_t$. A buyer who is removed in period T prior to purchase obtains utility $-\sum_{t=0}^{T}\delta^t y_t$. The discount factor, δ, is less than one.

3. EQUILIBRIUM WITHIN A PERIOD

A symmetric CSCE can be identified by equilibria within a period in the following way. Suppose that a given mechanism m^* is proposed as a candidate for an equilibrium within a period. Let m be any other mechanism, and suppose one seller deviates and plays this other mechanism. Given these mechanisms, and the profits from being a buyer in future periods, the buyers must make optimal participation choices and reports to the mechanism in which they choose to participate, should they choose to participate in any mechanism. These participation choices and reporting rules can be considered to be an "equilibrium for the buyers." Once the behavior of the buyers is established, the profitability of the deviation to the deviant can be assessed, bearing in mind that a CSCE presumes that the seller ignores his influence on the profits offered to buyers by other sellers. A symmetric CSCE in a period is a mechanism in which no deviation is profitable. Thus, our analysis starts with an analysis of the buyer behavior, and characterizes the reaction of buyers to a given seller's deviation, and then proceeds to the payoff to the deviating seller. A useful notion in this regard is a *CSCE best response*, which is a mechanism that maximizes a deviant seller's expected profits, given a candidate equilibrium mechanism and the CSCE assumptions.

By Remark 1, there will be a well defined function π_{t+1}^* which provides the value of being a buyer in the period $t+1$; that is, a buyer with value x will

[7] From a *Revelation Principle* perspective, the set of mechanisms is too small, because buyers are informed not only about their own value, but also about the mechanisms employed by other sellers, and a seller could design a mechanism to ask buyers to report on the mechanisms offered by other sellers in the current period. In an earlier draft of this paper, McAfee (1988), it is argued that pure strategy equilibria in direct mechanisms are equilibria to the game with larger strategy spaces for sellers. This occurs because a deviating seller believes that other sellers are not deviating, and thus the deviating seller does not expect to learn anything from asking buyers about the mechanisms employed by other sellers. However, the expansion of the sellers' strategy space could introduce other equilibria. It is also quite difficult to define the larger strategy space, since mechanisms must map the set of mechanisms into outcomes, leading to an infinite regress.

expect utility $\delta\pi^*_{t+1}(x)$ if he does not obtain the good in period t.[8] In the current period, suppose that $m_t - 1$ sellers are using one mechanism and the remaining seller, say seller 1, is using a second mechanism, and there are n_t buyers. I will look for symmetric CSCE for buyer behavior, that is, all buyers of a given type employ the same, possibly mixed, strategy. I suppose that buyers' valuations are independently and identically distributed in period t according to the distribution function G, which has density g.

If all buyers of the same type employ the same strategy in period t, there will be a function θ, so that $\theta(x)$ is the probability that a buyer with valuation x participates in seller 1's mechanism in period t. Because $\theta(x)$ is a probability:

$$(2) \qquad (\forall x) \quad \theta(x) \in [0,1].$$

In keeping with symmetry, suppose that buyers of type x go to any seller other than 1 with probability $(1-\theta(x))/(m_t - 1)$, provided that the candidate equilibrium mechanisms offer buyers nonnegative rents when they participate alone.[9] The function θ is, of course, a feature of the equilibrium, in that it is determined by buyer maximization. Before proceeding with this maximization, it is useful to introduce *reduced form mechanisms*.

A reduced form mechanism is a pair of functions (q, p), q: $[0,1] \to [0,1]$ and p: $[0,1] \to \mathbb{R}$ so that $q(y)$ is a buyer's probability of obtaining the item if he reports a value of y and $p(y)$ is the expected payment in this circumstance. The assumption that sellers use anonymous mechanisms insures that the reduced form mechanisms don't depend on the buyers' identities. Reduced form mechanisms incorporate two subtleties in this environment. First, the distribution of the number of buyers participating as well as the distribution of types for those buyers who do participate both depend on the buyers' participation strategies, which are endogenous. In principle, these distributions are determined by the function θ. However, in accounting for those buyers who do not participate at all, it is convenient to introduce the notation $z(0)$ for the probability that a buyer does not go to seller i. The relation between $z(0)$ and the function θ is explained below in condition (12). If seller 1 uses mechanism $\{(Q_n, P_n)\}_{n=1}^{n_t}$ then q is given by

$$\begin{aligned}(3) \qquad q(y) &= E_{-i}Q(y, x_{-i}) \\ &= \sum_{k=0}^{n_t-1} \binom{n_t-1}{k} z(0)^{n_t-k-1} \\ &\quad \times \int_0^1 \cdots \int_0^1 Q_{k+1}(y, x_1, \ldots, x_k)\theta(x_1)g(x_1) \\ &\quad \times \cdots \theta(x_k)g(x_k)\, dx_1 \cdots dx_k.\end{aligned}$$

[8] The discounting embodied in δ incorporates the risk of exogenous termination by nature; that is, δ includes both the probability that the agent is removed and the agent's pure time preference. Provided that the agent prefers current consumption over future consumption, we have $\delta \leqslant \alpha$.

[9] The case where buyers do not participate in the nondeviant mechanisms is analogous.

In a similar way, if the other $m_t - 1$ sellers use the mechanism $\{(Q_n^*, P_n^*)\}_{n=1}^{n_t}$ then q^* is given by:

$$(4) \qquad q^*(y) = E_{-i}Q^*(y, x_{-i}) = \sum_{k=0}^{n_t-1} \binom{n_t - 1}{k} \left(1 - \frac{z(0)}{m_t - 1}\right)^{n_t-k-1} \times \int_0^1 \cdots \int_0^1 Q_{k+1}^*(y, x_1, \ldots, x_k) \frac{1-\theta(x_1)}{m_t - 1} g(x_2) \times \cdots \frac{1-\theta(x_k)}{m_t - 1} g(x_k)\, dx_1 \cdots dx_k.$$

This is subtle because the value of θ itself is an equilibrium value that depends on profits offered by the sellers, which in turn are characterized in terms of the reduced form equations.

The second subtlety involves the incentive constraints. Without loss of generality, the deviator's mechanism may be assumed to be incentive compatible. This occurs because if the deviator offers a mechanism $\{(Q_n, P_n)\}_{n=1}^{n_t}$, which leads to buyer participation of θ and a reporting rule to the mechanism of y, so that a type x buyer reports $y(x)$, then the mechanism which offers the composition of $\{(Q_n, P_n)\}_{n=1}^{n_t}$ with y,[10] will have an equilibrium with the same participation function θ and honest reports. However, even if the other mechanisms are incentive compatible in equilibrium, the deviation of one seller will generally destroy the incentive compatibility of the other mechanism, because the buyers know that the anticipated equilibrium participation rate did not arise and adjust their reports accordingly. Let y^* refer to the equilibrium reporting rule for the other nondeviating mechanisms. While a deviation may alter the reporting used at other mechanisms, CSCE condition (i) posits that a deviating seller neglects the effect of his mechanism choice on profits available to buyers participating in other sellers' mechanisms.

Let $q(y), (q^*(y))$ be the probability that a buyer reporting the value y obtains the item at seller 1 (or any other seller). Variables without superscript * will refer to seller 1 in the current period t, while variables with the superscript * will refer to other sellers in the current period t, and subsequently to steady state values. Future values will be subscripted with $t+1$. A buyer who participates in one of the mechanisms offered by sellers other than 1 obtains an expected utility of

$$(5) \qquad \pi_t^*(x) = \max_y xq^*(y) + \delta\pi_{t+1}^*(x)(1 - q^*(y)) - E_{-i}P^*(y, x_{-i}),$$

where $E_{-i}P^*$ is the expected payment given participation in the seller's mechanism, defined analogously to $E_{-i}Q^*$ in (4). Similarly, if this buyer

[10] The composition replaces x_i with $y(x_i)$.

participates in seller 1's mechanism, the buyer expects profits of

$$\pi_t(x) = \max_y xq(y) + \delta\pi^*_{t+1}(x)(1-q(y)) - E_{-i}P(y, x_{-i}). \tag{6}$$

By the envelope theorem,

$$\pi'_t(x) = q(x) + \delta\pi^{*\prime}_{t+1}(x)(1-q(x)), \tag{7}$$

$$\pi^{*\prime}_t(x) = q^*(y^*(x)) + \delta\pi^{*\prime}_{t+1}(x)(1-q^*(y^*(x))). \tag{8}$$

Buyers choose the mechanism that offers the highest rents. Thus, CSCE requires

$$\theta(x) = \begin{cases} 0 & \text{if } \pi_t(x) < \pi^*_t(x), \\ 1 & \text{if } \pi_t(x) > \pi^*_t(x), \end{cases} \tag{9}$$

to hold; that is, buyers go to seller 1 if seller 1 is offering higher profits, go to another seller if seller 1 is offering smaller profits. If $\pi_t(x) = \pi^*_t(x)$, then a buyer with value x can go to seller 1 with any probability $\theta(x) \in [0,1]$. It is important to bear in mind that π_t and π^*_t depend on θ. But if (9) holds, no buyer can gain using a different strategy in period t.

In view of (8), a natural assumption on π^*_{t+1} is

$$(\forall x) \quad 0 \leqslant \pi^{*\prime}_{t+1}(x) \leqslant 1, \tag{10}$$

which will be satisfied provided that π^*_{t+1} is determined by mechanisms in the same way that π^*_t is, since $\pi^{*\prime}_t(x)$ is a convex combination of 1 and $\delta\pi^{*\prime}_{t+1}(x)$. Inequality (10) yields

$$(\forall x) \quad 0 \leqslant \delta\pi^{*\prime}_{t+1}(x) \leqslant \pi^{*\prime}_t(x) \leqslant 1. \tag{11}$$

I now turn to the optimization problem of seller 1 when all other sellers are using a fixed mechanism $\{(Q^*_n, P^*_n)\}^{n_t}_{n=1}$. In a CSCE, seller 1 takes his future profits as a seller, and the buyer profits, both in the future and in the present period, as fixed, although he allows for the endogeneity of the participation frequency. I denote the value of being a seller in the period $t+1$ by $\delta\Phi^*$. Given the reduced form mechanisms (q, p) and (q^*, p^*) for seller 1 and the other sellers respectively, π_t and π^*_t satisfy (5)–(8) and θ satisfies (2) and (9).

The function z given by:

$$z(x) = 1 - \int_x^1 \theta(s) g(s)\, ds \tag{12}$$

will play an important role in the analysis. Note that $z(x)$ is the probability that a given buyer either does not go to seller 1 or has a value less than x. In particular, $1 - z(0)$ is the probability that a buyer goes to seller 1 from seller 1's perspective, that is, not knowing the buyer's value. Similarly, $z(0)/(m_t - 1)$ is the probability a buyer goes to seller $i \geqslant 2$.

Because the seller has at most one unit to sell, there is a restriction on the reduced form probability q of receiving the good that emerges from (1) and (3). This restriction has been analyzed by Matthews (1984) and Border (1989) and is known as the implementability condition.[11] It is given by:

$$(13) \qquad (\forall x \in [0,1]) \quad \int_x^1 q(s) z'(s)\, ds \leqslant \frac{1}{n_t}\left(1 - z(x)^{n_t}\right).$$

The implementability condition captures the restriction that the seller has a single unit to sell. For example, the seller cannot promise to give the good with certainty to any potential buyer who reports a value in the interval $[y, 1]$ unless no buyers are expected from this interval, that is, θ is zero a.e. on this interval. This is because such a plan to set $q = 1$ would imply with positive probability that two agents with values in excess of y participate, and both expect the item with certainty, which is not feasible.

The implementability condition (13) is satisfied with equality when $q = z^{n_t - 1}$. This means that an agent receives the good if and only if no other agent with a higher value participates in the mechanism. This is tantamount to saying the seller holds an auction, because an auction awards the good to the highest value participant, provided that value exceeds a reserve price r.[12]

Using (12) and (6), I may express seller 1's expected profits as:

$$\begin{aligned}(14) \qquad \Phi &= \delta\Phi^* + n_t \int_0^1 (p(x) - \delta\Phi^* q(x))\theta(x) g(x)\, dx \\ &= \delta\Phi^* + n_t \int_0^1 [xq(x) + \delta\pi_{t+1}^*(x)(1 - q(x)) \\ &\qquad - \pi_t(x) - \delta\Phi^* q(x)] z'(x)\, dx \\ &= \delta\Phi^* + n_t \int_0^1 [(x - \delta\Phi^* - \delta\pi_{t+1}^*(x)) q(x) \\ &\qquad + \delta\pi_{t+1}^*(x) - \pi_t(x)] z'(x)\, dx.\end{aligned}$$

Seller 1 chooses his mechanism to maximize (14) subject to (2), (6), (7), (9), and (13). It is useful to introduce the notation:

$$(15) \qquad z_o(x) = \left(\frac{\pi_t^{*\prime}(x) - \delta\pi_{t+1}^{*\prime}(x)}{1 - \delta\pi_{t+1}^{*\prime}(x)}\right)^{1/(n_t - 1)},$$

$$(16) \qquad r^* = \sup\{r : \pi_t^*(r) = 0\}, \quad \text{and},$$

$$(17) \qquad r_{t+1}^* = \sup\{r : \pi_{t+1}^*(r) = 0\},$$

for ease in stating the theorem. To interpret z_o, consider a buyer who can go either to our seller or to another seller, and the second seller offers a probability $q^*(y^*(x))$ that a buyer with valuation x receives the item. Also suppose that the value of not purchasing from this second seller is $\delta\pi_{t+1}^*(x)$. Solving equation (8) for q^* shows that $q^* = z_0^{n_t - 1}$, that is, $z_0^{n_t - 1}$ is the probability that a buyer

[11] Both papers presume that the mechanism treats two buyers with the same valuation symmetrically, that is, the mechanism is anonymous. This is the motivation for restricting the seller to an anonymous mechanism. The fact that q is nondecreasing, a consequence of incentive compatibility and (6), is required to obtain (13) from Matthews' result. There is an analogous condition for q^* which plays no role in the analysis.

[12] Because of risk neutrality, many different mechanisms produce the same equilibrium allocation and expected profits. We refer to the seller as *holding an auction* when a standard auction will implement the allocation of the good and the expected payoffs.

with valuation x receives the good if he participates in the outside option and not in seller 1's mechanism. Bear in mind that while π_t^* depends on the equilibrium values y^* and θ, which in turn depend on seller 1's mechanism, it is assumed by CSCE condition (i) that seller 1 neglects this dependence.

THEOREM 1: *Suppose* (10) *holds, and assume*

(18) $\delta \Phi^* = r^* \leqslant r_{t+1}^*$,

(19) $\pi^{*\prime}(1) = \pi_{t+1}^{*\prime}(1) = 1$, *and*

(20) $(\forall x \in [r^*, 1]) \quad 0 \leqslant z_o'(x) \leqslant g(x)$.

Then, any mechanism which maximizes (14) *subject to* (2), (6), (7), (9), (13) *and the CSCE conditions satisfies*

(21) $\pi_t = \pi_t^*$,

(22) $z = z_o$, *and*

(23) $(\forall x \in [r^*, 1]) q(x) = z_o(x)^{n_t - 1}$.

Moreover, the CSCE best response can be implemented using a second price auction with reserve price of r^*.

PROOF: Observe that

$$(24) \qquad \begin{aligned} \Phi &\leqslant \delta\Phi^* + n_t \int_0^1 \big[(x - \delta\Phi^* - \delta\pi_{t+1}^*(x)) q(x) \\ &\qquad\qquad + \delta\pi_{t+1}^*(x) - \pi_t^*(x) \big] z'(x)\, dx \\ &\leqslant r^* + n_t \int_{r^*}^1 \big[(x - r^* - \delta\pi_{t+1}^*(x)) q(x) \\ &\qquad\qquad + \delta\pi_{t+1}^*(x) - \pi_t^*(x) \big] z'(x)\, dx \\ &= r^* + n_t \int_{r^*}^1 (1 - \delta\pi_{t+1}^{*\prime}(x)) \int_x^1 q(s) z'(s)\, ds \\ &\qquad - (\delta\pi_{t+1}^{*\prime}(x) - \pi_t^{*\prime}(x))(1 - z(x))\, dx \\ &\leqslant r^* + n_t \int_{r^*}^1 (1 - \delta\pi_{t+1}^{*\prime}(x)) \frac{1}{n_t} (1 - z(x)^{n_t}) \\ &\qquad - (\delta\pi_{t+1}^{*\prime}(x) - \pi_t^{*\prime}(x))(1 - z(x))\, dx \\ &\leqslant r^* + n_t \int_{r^*}^1 (1 - \delta\pi_{t+1}^{*\prime}(x)) \frac{1}{n_t} (1 - z_o(x)^{n_t}) \\ &\qquad - (\delta\pi_{t+1}^{*\prime}(x) - \pi_t^{*\prime}(x))(1 - z_o(x))\, dx. \end{aligned}$$

The first line follows from (9), since $\pi_t^*(x) > \pi_t(x)$ implies $z'(x) = \theta(x) g(x) = 0$. The second line follows form (11) and (16)–(18). The equality follows from integration by parts, noting $z(1) = 1$. The penultimate line uses (13) and (11), and the last inequality maximizes pointwise over z to obtain $z = z_o$.

The last line of (24) provides an upper bound on the seller's profits. This upper bound is achieved when (21)–(23) hold, and (13) holds with equality. The condition required for (13) to hold with equality, given (23), is $z_o(1) = 1$, or (19).

Moreover, (21)–(23) are feasible if they satisfy (2) and (9). By (21), (9) is vacuous. Inequality (20) is a restatement of (2), since $z'(x) = \theta(x)g(x)$. Thus, (18)–(20) insure that (21)–(23) solve the seller's maximization problem.

To see that an auction with reserve r^* maximizes the seller's revenue, note that (23) insures that any buyer with value x participating in the seller's mechanism obtains the good with probability equal to $z(x)^{n_t - 1}$, which is the probability that no higher value type participates in the mechanism. Thus, the highest value participant receives the item, and one implementation is for the seller to hold an auction. The reserve price is just the highest zero profit valuation, r^*. *Q.E.D.*

Theorem 1 admits a wide class of profit functions for which it is a CSCE best response for the seller to hold an auction, that is, sell to the highest value buyer participating in the mechanism. Intuitively, the reason is as follows. The constraint (19) forces, by (15), that $z_o(1) = 1$. Therefore, integrating (20), $G(x) \leqslant z_o(x) \leqslant 1$. By (15), $1 \geqslant \pi^{*\prime}(x) \geqslant \delta\pi^{*\prime}_{t+1}(x) + G(x)^{n_t - 1}(1 - \delta\pi^{*\prime}_{t+1}(x))$. When combined with (18), this bounds π^* below sufficiently that the constraint $\pi(x) \geqslant \pi^*(x)$ for $\theta(x) > 0$ binds everywhere. This, when combined with (11), insures that the seller wishes to sell to the highest value type, since (11) insures that the seller's payoff is nondecreasing in the buyer's valuation. But selling to the highest valuation buyer amounts to holding an auction.

Theorem 1 does not require a hazard rate assumption on the distribution of buyer values, which is required for the standard monopoly auction result. The reason for this difference concerns the exogeneity of profits in Theorem 1. In the standard result, the use of the hazard rate is to prevent the seller from attempting to reduce buyer profits for all high type buyers by reducing the probability that a given type buyer obtains the item. Consider a buyer with valuation x. In the standard result, reducing the profits of a type x buyer, by reducing $q(x)$, reduces the seller's payoff if the type x buyer appears, with density $g(x)$, but it reduces profits of all higher type buyers, who appear with probability $1 - G(x)$. This illustrates why an assumption on $(1 - G(x))/g(x)$ is useful in the standard result. In contrast, the seller is explicitly prohibited from reducing buyer profits in Theorem 1, which intuitively is why a hazard rate assumption plays no role.

I will now show that auctions are the only mechanism that can be used in a symmetric CSCE. In fact, a condition called self-replication, which is weaker than symmetry, is sufficient to guarantee that all sellers use auctions. A profit function π^* is said to be *self-replicating* if the best response of a seller to this profit function, that is, the mechanism which maximizes (14) subject to (2), (6), (7), (9), (13) and the CSCE conditions, induces expected profits for the participating buyers of $\pi = \pi^*$, and if $\theta(x) \in (0,1)$ for all x such that $\pi^*(x) > 0$. Self-replication of the induced profits is necessary for a mechanism to be part of a symmetric equilibrium in mechanisms, because the mechanisms must induce the same profits for the bidders, and share the bidders equally, in order to comprise a symmetric equilibrium. Self-replication is not sufficient to produce a symmetric equilibrium, because it does not require $\theta = 1/m$, nor does it require that the sellers use the same mechanism to induce those profits. The next result shows that, in this environment, the only self-replicating profit functions are those induced by auctions.

THEOREM 2: *If $r^* \leqslant r^*_{t+1}$ and* (10) *holds, then the only self-replicating profit functions in any CSCE are those induced by auctions with reserve price $r^* = \delta\Phi^*$.*

PROOF: Suppose π^* is self-replicating. By (7),

$$\pi_t^{*\prime}(x) = \pi_t'(x) = q(x) + \delta\pi_{t+1}^{*\prime}(x)(1 - q(x)).$$

By (15), $q(x) = z_o(x)^{n_t - 1}$. Thus, from (14),

$$\Phi = \delta\Phi^* + n_t\int_0^1 \big[(x - \delta\Phi^* - \delta\pi^*_{t+1}(x))q(x) + \delta\pi^*_{t+1}(x) - \pi^*_t(x)\big] z'(x)\, dx.$$

Now $x < r^*$ implies $q(x) = 0$ by (8) and (16). If $r^* > \delta\Phi^*$, then the seller can increase Φ by setting $q(x) > 0$ for $x \in (\delta\Phi^*, r^*)$. If $r^* < \delta\Phi^*$, then the seller can increase Φ by setting $q(x) = 0$ for $x \in (r^*, \delta\Phi^*)$. Thus self-replication implies that $r^* = \delta\Phi^*$. Therefore

$$\begin{aligned}
\Phi &= r^* + n_t\int_{r^*}^1 \big[(x - r^* - \delta\pi^*_{t+1}(x))q(x) + \delta\pi^*_{t+1}(x) - \pi^*_t(x)\big] z'(x)\, dx \\
&= r^* + n_t\int_{r^*}^1 (1 - \delta\pi^{*\prime}_{t+1}(x))\int_x^1 q(s) z'(s)\, ds - (\delta\pi^{*\prime}_{t+1}(x) - \pi^{*\prime}_t(x))(1 - z(x))\, dx \\
&= r^* + n_t\int_{r^*}^1 (1 - \delta\pi^{*\prime}_{t+1}(x)) \times \left(\int_x^1 z_o(s)^{n_t - 1} z'(s)\, ds - z_o(x)^{n_t - 1}(1 - z(x))\right) dx \\
&= r^* + n_t(n_t - 1)\int_{r^*}^1 (1 - \delta\pi^{*\prime}_{t+1}(x)) \times \left(\int_x^1 z_o(s)^{n_t - 2} z_o'(s)(1 - z(s))\, ds\right) dx.
\end{aligned}$$

The constraint $\theta \in (0,1)$ insures that $0 \leqslant z'(x) \leqslant g(x)$ does not bind. Therefore, the seller will maximize Φ by minimizing $z(x)$, and the feasibility constraint (13) binds everywhere. This forces the seller to hold an auction. *Q.E.D.*

Self-replication is a necessary condition for symmetry. If the profits π^* are determined by other sellers using some mechanism, and the remaining seller finds the same mechanism a best response, then the induced profits should be self-replicating. The fact that each seller offers the same mechanism guarantees the induced profits are the same across sellers, and each seller will share the agents equally, so $\theta(x) \in (0,1)$. Theorem 2 thus has the interpretation that the only symmetric CSCEs among mechanisms involve auction mechanisms, that is, mechanisms in which the highest valuation participant obtains the item. Auction mechanisms are, of course, a large class of mechanisms and what is ultimately shown by Theorem 2 is that, in symmetric CSCEs, sellers do not randomize, and sellers use an efficient reserve price, equal to the value of not selling. The major implications of Theorems 1 and 2 are summarized in the following remark.

REMARK 2: Theorem 2 shows that, in any symmetric CSCE, all sellers use auctions with reserve price r^*. Moreover, auctions with reserve satisfying

$$r^* = \delta \Phi^* \tag{25}$$

comprise a CSCE (it is readily verified that (10) and (18)–(20) are satisfied for the auction case, so that Theorem 1 guarantees it). In this case,[13] $\theta(x) = 1/m_t$ and $q^*(x)$ is just the probability that no higher value agent participates in the seller's mechanism, which is

$$z(x)^{n_t - 1} = \left(1 - \int_x^1 \frac{1}{m_t} g(s)\, ds\right)^{n_t - 1} = \left(1 - \frac{1 - G(x)}{m_t}\right)^{n_t - 1}. \tag{26}$$

In a steady state, $\pi_t^* = \pi_{t+1}^* = \pi^*$, and by (8) and (26)

$$\pi^{*\prime}(x) = z(x)^{n_t - 1} + \delta \pi^{*\prime}(x)\left(1 - z(x)^{n_t - 1}\right) \tag{27}$$

holds. Since r^* is a reserve price,

$$\pi^*(r^*) = 0. \tag{28}$$

I now turn to the steady state behavior of large economies.

4. THE STEADY STATE CSCE

Theorems 1 and 2 state that, under the CSCE assumptions that agents neglect certain strategic repercussions of their actions, an auction is a best response to a wide class of possible equilibrium candidate mechanisms, which includes auctions themselves, and that an auction with $r^* = \delta \Phi^*$ is the only candidate for a symmetric CSCE. This section will examine the steady state behavior of large economies of sellers holding auctions.

There are two independent uses of examining large economies. First the assumption that sellers neglect their own effect on buyer profits is unreasonable

[13] We are selecting the symmetric buyer participation equilibrium when sellers use identical strategies. Generally there will be other buyer participation equilibria. These other equilibria do not appear to lead to symmetric equilibria for sellers. The difficulty appears to be that an asymmetric buyer participation equilibrium tends to induce sellers to try to extract more surplus from the buyers than other sellers are extracting, because it is more costly to go to alternative sellers, as the buyers are not indifferent in asymmetric equilibria.

in the finite economy. However, as the economy grows, the ability of a single seller to affect profits available at other sellers' mechanisms goes to zero.[14] Second, the equations of motion of the finite economy are quite complex, because of the variability of the types of buyers, and thus the optimal reserve price will typically be random. This randomness vanishes in the limit.

There has been extensive discussion in the bargaining literature concerning the proper way to define limit economies, and various technical difficulties can arise.[15] Most of the difficulties concern the existence and properties of various matching processes with infinitely many agents. In particular, the strategy of a buyer, to randomize equally over the m sellers, does not have a well-defined limit strategy as m diverges. To avoid these difficulties, I will consider first the situation where finitely many buyers and sellers enter the economy in each period. The matching process then merely involves buyers randomizing over the stock of sellers, and buyers (and sellers) are removed either by buying (selling) or by the exogenous removal process. As the number of buyers and sellers is increased, the stochastic process governing the number of buyers and sellers converges to a deterministic process. This process, in turn, itself converges to a unique steady state of the distribution of buyer types. Thus, the problems of infinite matching processes are avoided because only the limit of a well-defined finite process is considered.

It should be stressed, however, that a CSCE does not involve optimal seller behavior in the finite environment, because any one seller can affect the buyer's profits at other sellers. In addition, in the finite environment, history will generally be relevant, because, for instance, a seller who fails to sell for a long period of time will deduce that the buyers in the system have unusually low valuations, and hence the seller will wish to lower his reserve price. In the limit as the number of agents diverges, however, there will exist a stationary distribution of buyer types, and such effects do not arise.

The concept of a large economy steady state will be formalized below, and involves two elements. First, I examine the equations of motion, governing the populations of buyers and sellers, of a large economy. In principle, these equations could be computed for any sequence of reserve prices employed by the sellers. However, only the case of a constant reserve price will be examined. In the limit, these equations of motion converge to deterministic equations, and these equations have a unique steady state. This provides a mapping from reserve prices to limiting steady state populations. The second element in the

[14] The most that buyer profits can be lowered occurs when a seller refuses to sell. Increasing buyer profits does not benefit the seller, and the only use of the invariance of buyer profits is in insuring that the seller cannot lower $\pi_i^*(x)$, which is used in the first line of (24). At most, a deviating seller can reduce the profits available at other sellers by not selling. This has the effect of reducing the number of sellers by one. In a steady state, the effect on the profits of buyers is of order $1/m$, where m is the number of sellers, and the number of buyers is kept proportional to the number of sellers. This demonstrates that, as the number of sellers grows, holding proportional the number of buyers, the effect any one seller can have on buyer profits goes to zero at rate $1/m$.

[15] See Gale (1986).

large economy steady state requires the sellers to be optimizing. By Remark 2, this amounts to the imposition of equation (25).

Consider a sequence of time periods $t = 0, 1, 2 \dots$. In each time period t, m_t sellers hold auctions with reserve price r^*. There are n_t buyers, and the population of buyers at time t may be viewed as independent draws from the c.d.f. G_t, which has a continuous density g_t. Buyers randomize over sellers; that is, a given buyer participates in seller i's auction with probability $1/m_t$.

A seller sells provided he obtains a buyer with valuation exceeding the reserve price. Thus, in any one period t, a seller fails to sell with probability $(1 - (1/m_t)(1 - G_t(r^*)))^{n}_t$, and otherwise sells. It follows that the expected value of m_{t+1} is:

$$(29) \qquad Em_{t+1} = \alpha m_t \left(1 - \frac{1}{m_t}(1 - G_t(r^*))\right)^{n_t} + S,$$

since $(1 - (1/m_t)(1 - G_t(r^*)))^{n_t}$ do not sell by auction, and α of these survive the exogenous termination, and then S new sellers enter.

For buyers with values less than the reserve price r^*, no exit occurs via auction. Thus

$$(30) \qquad En_{t+1}G_{t+1}(x) = \alpha n_t G_t(x) + bSF(x) \quad \text{for } x < r^*.$$

A buyer with value $y > r^*$ wins an auction with probability $(1 - (1/m_t)(1 - G_t(y)))^{n_t - 1}$. Thus, taking expectations over values $y > x$, buyers with values randomly drawn subject to exceeding x win auctions with probability:

$$(31) \qquad \int_x^1 \left(1 - \frac{1}{m_t}(1 - G_t(y))\right)^{n_t - 1} \frac{g_t(y)}{1 - G_t(x)}\, dy = \frac{m_t}{n_t(1 - G_t(x))}\left[1 - \left(1 - \frac{1}{m_t}(1 - G_t(x))\right)^{n_t}\right].$$

Otherwise, the buyer remains in the stock, still with a value in excess of x. By (31), therefore, for $x > r^*$,

$$(32) \qquad En_{t+1}(1 - G_{t+1}(x)) = \alpha n_t(1 - G_t(x))\left[1 - \frac{m_t}{n_t(1 - G_t(x))} \times \left(1 - \left(1 - \frac{1}{m_t}(1 - G_t(x))\right)^{n_t}\right)\right] + bS(1 - F(x))$$
$$= \alpha n_t(1 - G_t(x)) - \alpha m_t\left[1 - \left(1 - \frac{1}{m_t}(1 - G_t(x))\right)^{n_t}\right] + bS(1 - F(x)).$$

Introduce the notation

(33) $$\mu_t(x)=\frac{n_t}{m_t}(1-G_t(x)),$$

(34) $$\sigma_t=\frac{m_t}{S},\quad \text{and}$$

(35) $$\beta_t=\mu_t(r^*).$$

Equations (33)–(35) permits us to rewrite (29) and (32) to yield:

(36) $$E\sigma_{t+1}=\alpha\sigma_t\left(1-\frac{\beta_t}{n_t}\right)^{n_t}+1\underset{S\to\infty}{\to}\alpha\sigma_t e^{-\beta_t}+1,$$

(37) $$E\sigma_{t+1}\mu_{t+1}(x)=\alpha\sigma_t\mu_t(x)-\alpha\sigma_t\left[1-\left(1-\frac{\mu_t(x)}{n_t}\right)^{n_t}\right]+b(1-F(x))$$
$$\underset{S\to\infty}{\to}\alpha\sigma_t\mu_t(x)-\alpha\sigma_t(1-e^{-\mu_t(x)})+b(1-F(x)).$$

As S gets large, both m_t and $n_t(1-G_t(x))$ approach binomials, and thus their variance is linear in S. As a result, σ_t and $\mu_t(x)$ converge almost surely, and in the limit, the expectation operator may be dropped from (36) and (37).[16] This gives the dynamic equations governing the numbers of agents as a proportion of the size of an entering cohort of sellers. I now show that there is a unique globally stable steady state.

LEMMA 3: *The process* (36)–(37) *is globally dynamically stable, with* $(\mu_t,\sigma_t,\beta_t)\to(\mu,\sigma,\beta)$ *satisfying*:

(38) $$\sigma=(1-\alpha e^{-\beta})^{-1},$$

(39) $$\beta=1+\frac{b(1-F(r^*))}{\sigma(1-\alpha)},\quad \textit{and}$$

(40) $$(1-\alpha)\mu(x)+\alpha(1-e^{-\mu(x)})=\frac{b(1-F(x))}{\sigma}.$$

Moreover, (38)–(40) *admit exactly one solution.*

The Proof is contained in the Appendix, Part II.

Lemma 3 demonstrates that, for any given reserve price r^*, there is an associated unique steady state distribution of buyer types G and a unique ratio β giving the number of active buyers, i.e. those with values exceeding r^*, to sellers. To complete the construction of the steady state CSCE, we need find an r^* which is a best response to G. By Remark 2, we need to know only the limiting value of Φ^* in the steady state. This is constructed using limiting values of equations (14), (26), and (27) for the finite case.

I define a large economy steady state CSCE to be a sextuple $(r^*,\mu,\pi^*,\Phi,\beta,\sigma)$ as follows. First, $\mu(x)$, β, and σ are the limiting values, as S

[16] A derivation is provided in part I of the Appendix.

diverges, of the number of buyers with values in excess of x per seller, the number of buyers with values in excess of r^* per seller, and the number of sellers in the population per entering seller, respectively. Second, π^* and Φ are the limiting values of the buyer profits and seller profits given in (27) and (14) respectively, where z is given its limiting value. Third, equation (25), which insures the auctions are a best response, holds.

Consider again the large finite case, so that there are m_t sellers and n_t buyers, with values drawn from the distribution function G, present in period t. For a buyer with value $x > r^*$, the probability of winning in the current period, when all sellers employ auctions, is, by (26) and (33),

$$z_o(x)^{n_t - 1} = \left(1 - \frac{1 - G(x)}{m_t}\right)^{n_t - 1} \to e^{-\mu(x)}. \tag{41}$$

Thus, by (27), at the steady state given in Lemma 3,

$$\lim_{s \to \infty} \pi^{*\prime}(x) = \frac{e^{-\mu(x)}}{1 - \delta(1 - e^{-\mu(x)})}. \tag{42}$$

Equation (42) yields

$$(1 - \delta)\pi^{*\prime}(x) = (1 - \delta\pi^{*\prime}(x))e^{-\mu(x)}. \tag{43}$$

If all sellers employ second price auctions with reserve r^* in all periods, and the number of sellers and the distribution of buyers satisfies the steady state given in Lemma 3, then a seller's present value of profits is given by

$$\begin{aligned} \Phi(r^*) &= r^* + n\int_{r^*}^{1}\Big[(x - r^* - \delta\pi^*(x))z_o(x)^{n-1} \\ &\qquad\qquad - (1-\delta)\pi^*(x)\Big]z_o'(x)\,dx \\ &\to r^* + \int_{r^*}^{1}\big[(x - r^* - \delta\pi^*(x))e^{-\mu(x)} \\ &\qquad\qquad - (1-\delta)\pi^*(x)\big](-\mu'(x))\,dx \\ &= r^* + (1 - r^* - \delta\pi^*(1)) - \int_{r^*}^{1}(1 - \delta\pi^{*\prime}(x))e^{-\mu(x)}\,dx \\ &\quad - (1-\delta)\int_{r^*}^{1}\pi^{*\prime}(x)\mu(x)\,dx \\ &= 1 - \delta\pi^*(1) - \int_{r^*}^{1}(1-\delta)\pi^{*\prime}(x)\,dx \\ &\quad - (1-\delta)\int_{r^*}^{1}\pi^{*\prime}(x)\mu(x)\,dx \\ &= 1 - \int_{r^*}^{1}\pi^{*\prime}(x)[1 + (1-\delta)\mu(x)]\,dx \\ &= 1 - \int_{r^*}^{1}\frac{e^{-\mu(x)}(1 + (1-\delta)\mu(x))}{1 - \delta + \delta e^{-\mu(x)}}\,dx. \end{aligned} \tag{44}$$

The first line restates (14), the second substitutes (41), that is, substituting $e^{-\mu(x)}$ for $z(x)^{n-1}$ and $-\mu'(x)$ for $nz'(x)$, the third integrates by parts, the fourth line employs (43), the fifth collects terms, and the sixth substitutes (42).

In the large economy, a seller who deviates to a mechanism other than the auction with reserve price r^* takes π^* as given, and π^* is constant over time at the steady state, which is why the time subscript has been suppressed. Note that $\pi = \pi^* = \pi^*_{t+1}$ satisfies equations (18)–(20), provided $r^* = \delta\Phi^*$. Thus, by Theorem 1, a seller cannot gain by deviating if the expected present value of being a seller in the subsequent period, which is $\delta\Phi(r^*)$, equals r^*. That is, if (25), (38)–(40), (42), and (44) all simultaneously hold, then $(r^*, \mu, \pi^*, \Phi, \beta, \sigma)$ is a large economy steady state CSCE, with all sellers holding auctions with reserve price r^*, and buyers mixing symmetrically over sellers.

THEOREM 4: *There is a unique value r^* so that* (25), (38)–(40), (42), *and* (44) *are jointly satisfied, and $r^* \in (0, \delta)$. That is, there is a unique large economy steady state* CSCE.

The proof is straightforward and is in part IV of the Appendix.

Theorem 4 shows that, in large economies with exogenous entry by buyers and sellers, there is an equilibrium where sellers hold auctions and buyers randomize over which auction they participate in. Sellers cannot unilaterally do better than holding an auction. In addition, each seller posts a reserve price equal to the value of not selling, in contrast to the usual monopoly result, in which the seller's reserve exceeds the value of not selling. The value of not selling in this environment is merely the present value of being a seller in the next period. In the usual monopoly model, the seller gains from the inefficient reserve price by driving down profits. In large economies, it is impossible for a seller to drive down the buyers' profits, because the set of alternatives available to the buyers is too large.

5. COMPARATIVE STATICS

In this environment, it is possible to increase the number of buyers per seller by increasing b, increase the exogenous removal rate $1-\alpha$, and increase the discount rate δ. The latter two experiments are not available in the standard monopoly auction model. Besides assessing the effect on the reserve price r^*, the effect on the ratio β of active buyers to sellers is also of interest. Derivations of the claims in this section are provided in part V of the Appendix.

Not surprisingly, increasing the number of entering buyers per seller, b, increases both the ratio of active buyers to sellers β and the equilibrium reserve price r^*. This is in contrast to the usual monopoly model, however, where the number of buyers per seller does not affect the reserve price.

Increasing the exogenous removal rate $1-\alpha$ has a complicated effect on r^*. However, it can be shown that, if $b \leqslant 1$, then increasing α decreases r^*. The effect is as follows. If $b \leqslant 1$, that is, there are fewer buyers entering than sellers, then increasing the proportion that survive each period tends to decrease the number of buyers per seller in the stock. The excess sellers tend to accumulate, and this disadvantages the sellers, and competition lowers their payoffs and hence the reserve price.

If $b > 1$, the situation is less clear. Suppose that $b(1 - F(r^*)) > 1$, that is, $\beta > 1$ by Lemma 4. Then the number of active buyers per seller exceeds 1, and increasing α tends to further increase the number of active buyers per seller. Counting this effect is a change in the equilibrium distribution of buyer valuations. High value buyers tend to purchase quickly, and thus the accumulation of buyers is mostly of low value buyers, tending to reduce seller payoffs. Thus the extra active buyers per seller may not increase enough to outweigh the reduction in average values. However, I have not found an example.

When δ is increased, r^* rises and β falls. This is not surprising. As δ rises, the future becomes more valuable to the seller, and r^* is merely the present value of being a seller in the future, so it must rise. Since r^* rises, there are fewer active buyers per seller entering, and β falls.

As a final comparative static exercise, consider the case when $\alpha = \delta$, that is, all discounting is due to the risk of exogenous termination. Since r^* tends to decrease in α, at least if $b(1 - F(r^*)) \leq 1$, and increase in δ, the effect on r^* is unclear. For uniform F, this case has a simple form, derived in the Appendix. Figures 1 and 2 illustrate the effect of changing b on r^* and β, while Figures 3 and 4 illustrate the effect of changing $\alpha = \delta$.

As δ goes to zero, r^* goes to zero as well, because the present value of being a seller goes to zero. However, $\Phi = r^*/\delta$ does not vanish, and indeed,

$$\lim_{\alpha = \delta \to 0} \Phi = b^{-1}\left[b - 2 + e^{-b}(b + 2)\right].$$

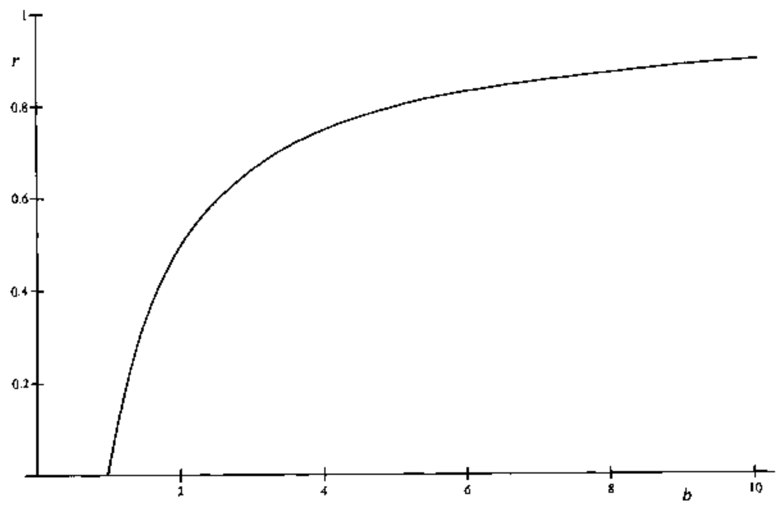

FIGURE 1.—r varying with b, $\alpha = \delta = 0.999$.

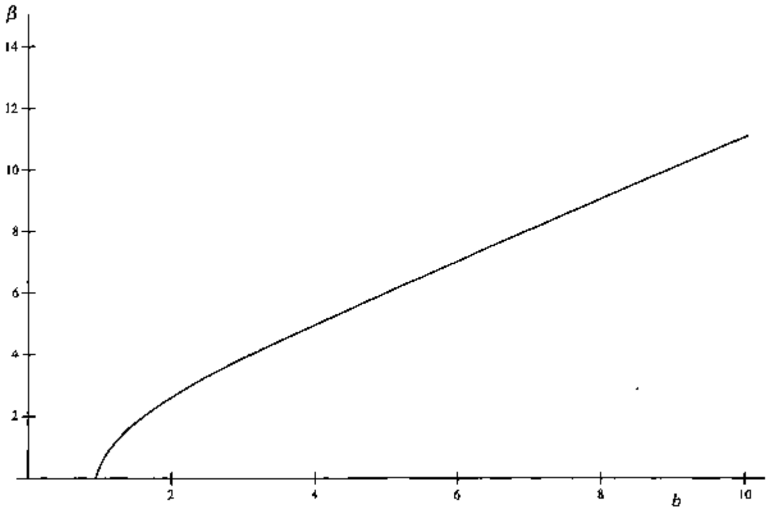

FIGURE 2.—β varying with b, $\alpha = \delta = 0.999$.

As $\alpha = \delta$ goes to one, neither sellers nor buyers are lost through attrition, but only though sales and purchases. As a result, either sellers, if $b < 1$, or buyers, if $b > 1$, accumulate. If $b < 1$, and sellers accumulate, r^* is driven to zero, because even though sellers are becoming more patient, they are becoming more numerous, and the competition forces the price down.

If $b > 1$, buyers tend to accumulate as $\alpha = \delta \to 1$, driving r^* up. In addition, as α gets close to 1, and if b is moderately large (e.g. over 5), β becomes approximately equal to $b + 1$, which is an upper bound for β, and r^* approaches $(b-1)/b$, and is exactly equal to this in the limit at $\alpha = 1$. Thus, when

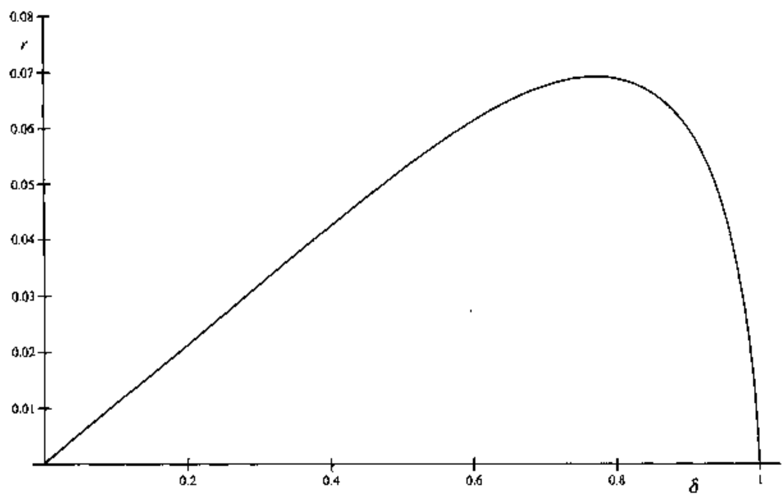

FIGURE 3.—r varying with δ, $\alpha = \delta$, $b = 1$.

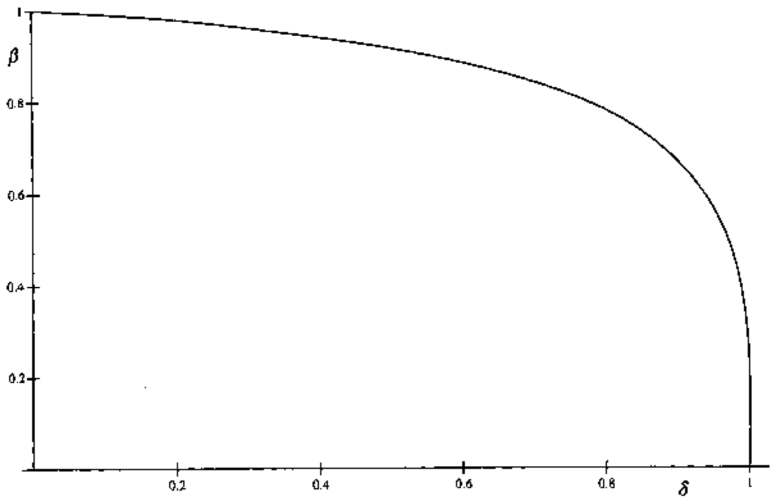

FIGURE 4.—β varying with δ, with $\alpha = \delta$, $b = 1$.

exogenous termination and discounting are eliminated from the model, the reserve price is zero if there are at least as many sellers as buyers entering, and otherwise it is one minus the ratio of sellers to buyers.[17]

The experiment of asking whether the reserve price is efficient, that is, does the reserve price maximize the sum of buyer and seller utilities, is easy to pose in this model. Consider an entering cohort. The present value of utility for this cohort is, per seller,

$$v = \Phi(r^*) + b\int_{r^*}^{1} \pi^*(x) f(x)\, dx.$$

Thus, one can ask whether the reserve price maximizes the present value of each cohort, since this is a constant.

As it turns out, the equilibrium does maximize the present value of a cohort if $\alpha = \delta$, and otherwise may not. In particular, if the equilibrium would have $\beta = 1$, which is possible, then the reserve price is too low if $\alpha > \delta$; that is, if all present and future sellers increased their reserve price, then once the steady state is reached, entering cohorts would enjoy higher utility.

One might intuitively expect the reserve price to be too low. The equilibrium reserve equals the value of being a seller in the next period. The first best reserve should equal the social value of postponing sale and having an extra good in the next period, which is the sum of the value of being a seller in the next period and the additional value accruing to buyers not extracted by the

[17] Interestingly, this is the outcome which a naive supply and demand analysis yields. Flow supply equals flow demand when $\beta = 1$, and for the uniform distribution and $\alpha = \delta = 1$, this occurs when $r^* = 1 - b^{-1}$.

seller. Since the latter is positive, one anticipates that reserve prices are too low, and this makes it somewhat mysterious why they are efficient if $\alpha = \delta$.[18]

6. CONCLUSION

This paper fails to address three very important aspects of the problem of mechanism design in the presence of competing mechanisms. First, a change in one seller's mechanism will tend to alter the distribution of buyers participating in the other mechanisms, thereby altering the buyers' expected payoffs from the other mechanisms. This effect was finessed by appealing to a limit economy. Second, the analysis focuses exclusively on steady states, and contributes nothing toward an analysis of mechanism design off the steady state. Third, sellers do not typically design auctions, but rather auctions are designed by third parties not present in this model.

Consider first the case of n buyers and m sellers in a one shot game. The sellers choose mechanisms, then the buyers choose which mechanism to participate in, the mechanisms are operated, and the game ends. It can be readily shown that, in such an environment, it is not an equilibrium for sellers to hold auctions with the same reserve price. Thus, the result of this paper is an equilibrium for a limit economy that will fail in all finite economies. In this regard, the results presented here can be viewed as the mechanism design analog to the theory of perfect competition. Obviously, a solution to the competing mechanism problem with finitely many, but more than one, mechanism designers would be of interest.

Even given the large economy assumption, the dynamics associated with the evolution of the system over time is also important, although there is a trivial special case ($\alpha = 1$) where no dynamics arise. The problem with dynamics is basically restricted to the lack of an analog of Theorem 1 when the reserve price varies with time. Note that Theorem 1 permits the bidder's expected profits to vary. In addition, the dynamic equations for β, G, n, and m are reasonably straightforward, so long as r^* is constant. However, once the environment is allowed to vary over time, there is reason to expect the equilibrium reserve price to vary. This, in turn, will render Theorem 1 inapplicable, although, if the reserve prices rise over time, Theorem 1 still applies. Equations (30), (36), and (37) can be used to characterize the evolution of the buyer types, with (25) providing reserve prices. There is still more work to be done, however, to establish what conditions on the initial distribution of buyer types μ will lead to reserve prices rising in *every* period, which is necessary to apply Theorem 1. Moreover, Theorem 1 sheds no light on the effect of a worsening over time of conditions for sellers. In any case, an analysis of the dynamics of r^*, or of the

[18] In an early version of this paper (1988) the case of endogenous entry of buyers was considered, and the equilibrium reserve price, which satisfied (25), was shown to be inefficiently low, even if $\alpha = \delta$. The intuition was clearer in that case, because sellers did not consider the effect of their reserve price on the equilibrium distribution of buyer types and on the entry of buyers.

equilibrium mechanism, would shed light on this model, and on competing mechanisms generally.

This paper falls far short of a real theory of equilibrium institutions partly because it places the design of institutions in the hands of the sellers. A more satisfactory approach requires explicit modelling of the role of intermediaries, or auctioners, who compete among each other for both buyers and sellers. Such a theory faces at least one major obstacle: it is advantageous to bring all the buyers and sellers together at one intermediary's location. No matter what the intermediary's mechanism looks like, buyers and sellers won't *unilaterally* deviate from the proposal to all go to a certain location. This observation may explain the "stickiness" of institution, or may only be an obstacle to a reasonable formulation of the design and evolution of selling institutions.

In spite of these defects, the model represents an alternative to the models of price formation based on bargaining, such as Rubinstein and Wolinsky (1985). Perhaps the most interesting feature of the model is the endogeneity of the matching process. Although the equilibrium matching process is akin to Rubinstein and Wolinsky's, sellers have the ability, by lowering reserve prices or generally making the mechanism more attractive to buyers, to attract more buyers, and their desire not to do so in equilibrium acts as a constraint on the system, affecting the determination of prices. Moreover, the mechanism by which prices are set is itself an equilibrium phenomenon, in that sellers are not constrained to hold auctions, but rather choose to, given the decision of other sellers to sell by auction. This is in contrast to the alternating offer bargaining model imposed by Rubinstein and Wolinsky, where both the matching technology and the transaction mechanism are exogenous.

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APPENDIX

PART I: *Justification for Removing the Expected Value Operator in Equations* (36) *and* (37)

Let τ_i be the binomial variable which is 1 if seller i does not sell and is not terminated, that is, seller i remains in the stock, and 0 otherwise.

$$E\tau_i=\alpha\left(1-\frac{1-G_t(r^*)}{m_t}\right)^{n_t}\approx \alpha e^{-\beta_t},$$

$$\text{for } i\neq j,\quad E\tau_i\tau_j=\alpha^2\left(1-\frac{2(1-G_t(r^*))}{m_t}\right)^{n_t}\approx \alpha^2 e^{-2\beta_t},$$

$$m_{t+1}=S+\sum_{i=1}^{m_t}\tau_i.$$

This yields:

$$Em_{t+1} = \alpha m_t e^{-\beta_t},$$

$$\begin{aligned}\operatorname{Var}(m_{t+1}) &= E\left(\sum_{i=1}^{m_t} \tau_i\right)^2 - \left(E\sum_{i=1}^{m_t} \tau_i\right)^2 \\ &= m_t E\tau_i^2 + m_t(m_t-1)E\tau_i\tau_j - m_t^2\alpha^2 e^{-2\beta_t} \\ &= m_t(\alpha e^{-\beta_t} - \alpha^2 e^{-2\beta_t}) \leqslant \frac{m_t}{4} \leqslant \frac{m_a + tS}{4}.\end{aligned}$$

Therefore, $\operatorname{var}(\sigma_{t+1}) \leqslant (m_a + tS)/4S^2 \to 0$ as $S \to \infty$.

Since m_t/S converges a.s. as $S \to \infty$, it follows that n_t/S converges as well, and thus that m_t/n_t converges, as desired.

PART II: *Proof of Lemma* 3

Using (36) and (37) with $x = r^*$, since $\beta_t = \mu_t(r^*)$,

$$\begin{aligned}\sigma_{t+1}\beta_{t+1} &= \alpha\sigma_t(\beta_t - 1) + \alpha\sigma_t e^{-\beta_t} + b(1 - F(r^*)) \\ &= \alpha\sigma_t(\beta_t - 1) + \sigma_{t+1} - 1 + b(1 - F(r^*)), \quad \text{or}\end{aligned}$$

$$\sigma_{t+1}(\beta_{t+1} - 1) = \alpha\sigma_t(\beta_t - 1) - 1 + b(1 - F(r^*)).$$

Therefore,

$$\text{(A1)} \qquad \sigma_t(\beta_t - 1) \to \frac{b(1 - F(r^*)) - 1}{1 - \alpha}.$$

For large t, using (36) and (A1),

$$\text{(A2)} \qquad \sigma_{t+1} \approx 1 + \alpha\sigma_t e^{-1-\gamma/\sigma_t},$$

where $\gamma = (b(1 - F(r^*)) - 1)/(1 - \alpha)$ and $\approx$ indicates an arbitrarily good approximation, since $\sigma_t(\beta_t - 1)$ is monotonic in t. Note that in (A2),

$$\frac{\partial\sigma_{t+1}}{\partial\sigma_t} \approx \alpha e^{-1-\gamma/\sigma_t}(1 + \gamma/\sigma_t) \leqslant \alpha e^{-1} < 1.$$

Thus σ_t converges to a value satisfying

$$\text{(A3)} \qquad \sigma = (1 - \alpha e^{-1-\gamma/\sigma})^{-1} = (1 - \alpha e^{-\beta_t})^{-1}.$$

The combination of (A1) and (A3) yield (38) and (39). Moreover, since σ_t converges, β_t converges to $1 + \gamma/\sigma$. Since $\sigma_t \to \sigma$, (37) yields (40). To see that (38)–(40) have a unique solution, let

$$\Psi(\beta) = (1 - \alpha)(\beta - 1) - (1 - \alpha e^{-\beta})[b(1 - F(r^*)) - 1].$$

Then (38) and (39) combine to show that $\Psi(\beta) = 0$. Therefore:

$$\Psi(0) = -(1 - \alpha)b(1 - F(r^*)) < 0,$$

$$\lim_{\beta\to\infty} \Psi(\beta) = \infty,$$

$$\Psi'(\beta) = (1 - \alpha) - \alpha e^{-\beta}[b(1 - F(r^*)) - 1],$$

$$\Psi''(\beta) = \alpha e^{-\beta}[b(1 - F(r^*)) - 1].$$

Thus,

$$\Psi'(\beta) = 0 \Rightarrow \Psi''(\beta) = 1 - \alpha > 0.$$

Hence, any extreme point is a minimum and since $\Psi(0) < 0$, there exists a unique solution to $\Psi(\beta) = 0$, as any two solutions would imply a maximum between them. The uniqueness of σ and μ is apparent from (38) and (40), once β is uniquely identified. *Q.E.D.*

I will need the following lemma in the proof of Theorem 4 and in part V.

PART III: *Lemma A*

LEMMA A: *Fix r^* and consider the global steady state of Lemma* 3.

$$\text{(A4)} \qquad \frac{\partial\beta}{\partial\alpha} \gtreqless \textit{ as } b(1 - F(r^*)) \gtreqless 1,$$

$$\text{(A5)} \qquad \frac{\partial\beta}{\partial b} > 0,$$

$$\text{(A6)} \qquad \frac{\partial\beta}{\partial r^*} < 0,$$

$$\text{(A7)} \qquad (\forall x \in (r^*, 1))\frac{\partial\mu(x)}{\partial r^*} < 0.$$

PROOF OF LEMMA A: It is useful to eliminate σ from (38)–(39) and work directly with

$$\text{(A8)} \qquad \Psi(\beta; \alpha, b, r^*) = (1-\alpha)(\beta - 1) - (1 - \alpha e^{-\beta})[b(1 - F(r^*)) - 1].$$

β is defined by $\Psi = 0$, and at this solution, $\partial\Psi/\partial\beta > 0$ from the proof of Lemma 3.

$$\text{(A9)} \qquad \Psi_\alpha = -(\beta - 1) + e^{-\beta}[b(1 - F(r^*)) - 1]$$
$$= -(\beta - 1) + e^{-\beta}\frac{(1-\alpha)(\beta - 1)}{1 - \alpha e^{-\beta}}$$
$$= -(\beta - 1)\frac{1 - e^{-\beta}}{1 - \alpha e^{-\beta}} \gtreqless 0 \text{ as } \beta \lesseqgtr 1.$$

Thus, we obtain (A4):

$$\frac{\partial\beta}{\partial\alpha} = -\frac{\Psi_\alpha}{\Psi_\beta} \gtreqless 0 \quad \text{as} \quad \beta \gtreqless 1.$$

$$\text{(A10)} \qquad \Psi_b = -(1 - \alpha e^{-\beta})(1 - F(r^*)) < 0,$$

and therefore yielding (A5): $(\partial\beta/\partial b) = -(\Psi_b/\Psi_\beta) > 0$. Finally, (A6) and (A7) follow from

$$\text{(A11)} \qquad \Psi_{r^*} = (1 - \alpha e^{-\beta}) b f(r^*) > 0 \Rightarrow \frac{\partial\beta}{\partial r^*} < 0.$$

From (40), substituting (38) to eliminate σ, we obtain (A7):

$$\text{(A12)} \qquad \frac{\partial\mu(x)}{\partial r^*} = \frac{b(1 - F(r^*))\alpha e^{-\beta}\dfrac{\partial\beta}{\partial r^*}}{1 - \alpha + \alpha e^{-\mu(x)}} < 0. \qquad \textit{Q.E.D.}$$

PART IV: *Proof of Theorem* 4

Define

$$\text{(A13)} \qquad v(r^*) = r^* - \delta\Phi(r^*) = r^* - \delta\left(1 - \int_{r^*}^{1} \frac{e^{-\mu(x)}(1 + (1-\delta)\mu(x))}{1 - \delta + \delta e^{-\mu(x)}}\,dx\right).$$

Let

$$\text{(A14)} \qquad \gamma(\mu)=\frac{e^{-\mu}(1+(1-\delta)\mu)}{1-\delta+\delta e^{-\mu}}\in[0,1].$$

Note that

$$\text{(A15)} \qquad \gamma'(\mu)=-\frac{(1-\delta)e^{-\mu}[(1-\delta)\mu+\delta(1-e^{-\mu})]}{(1-\delta+\delta e^{-\mu})^2}<0.$$

Now

$$\nu(0)=-\delta\left(1-\int_0^1\gamma(\mu(x))\,dx\right)=-\delta\int_0^1 1-\gamma(\mu(x))\,dx<0 \quad \text{and}$$

$$\nu(\delta)=\delta\int_0^1\gamma(\mu(x))\,dx>0.$$

To reduce notational clutter, the dependence of μ on β is suppressed. Note that, by (40), μ depends on r^* only through the dependence of β on r^*. ν is obviously continuous, so a solution to $\nu(r^*)=0$ exists in $(0,\delta)$. Moreover,

$$\text{(A16)} \qquad \nu'(r^*)=1-\delta\gamma(\mu(r^*))+\delta\int_{r^*}^1\gamma'(\mu(x))\frac{\partial\mu(x)}{\partial r^*}\,dx>0,$$

by (A7), (A14), and (A15). Therefore the solution is unique. *Q.E.D.*

PART V: *Comparative Statics Derivations for Section 5*

In addition to the functions Ψ and ν given in (A8) and (A13), we shall use

$$\text{(A17)} \qquad (1-\alpha)\mu(x)+\alpha(1-e^{-\mu(x)})=\frac{1-F(x)}{1-F(r^*)}\left[(1-\alpha)\beta+\alpha(1-e^{-\beta})\right],$$

which is a consequence of (38)–(40). We have from the proof of Lemma 3 that, at $\Psi(\beta)=0$,

$$\text{(A18)} \qquad \frac{\partial\Psi}{\partial\beta}>0.$$

V.1—*Derivatives with respect to b*: From (A17), $\partial\mu(x)/\partial\beta>0$. Thus, since b does not appear in ν,

$$0=\nu_{r^*}\,dr^*+\nu_\beta\,d\beta$$

$$=\nu_{r^*}\,dr^*+\delta\int_{r^*}^1\gamma'(\mu(x))\frac{\partial\mu(x)}{\partial\beta}\,dx\,d\beta.$$

By (A15) and (A16), dr^* and $d\beta$ have the same sign. Moreover,

$$0=\Psi_{r^*}\,dr^*+\Psi_\beta\,d\beta+\Psi_b\,db.$$

By (A10), (A11), and (A18), we can conclude that

$$\frac{dr^*}{db}>0 \quad \text{and} \quad \frac{d\beta}{db}>0.$$

V.2—*Derivatives with respect to α for $\beta\leqslant 1$*: Observing that the function

$$\frac{y-1+e^{-y}}{(1-\alpha)y+\alpha(1-e^{-y})}$$

is increasing in y, and $\mu(x) \leqslant \beta$ by (A17) for $x \geqslant r^*$, we have

$$\frac{\mu(x) - 1 + e^{-\mu(x)}}{(1-\alpha)\mu(x) + \alpha(1 - e^{-\mu(x)})} \leqslant \frac{\beta - 1 + e^{-\beta}}{(1-\alpha)\beta + \alpha(1-e^{-\beta})}$$

which implies by (A17)

$$\mu(x) - 1 + e^{-\mu(x)} \leqslant \frac{1 - F(x)}{1 - F(r^*)}(\beta - 1 + e^{-\beta}),$$

which implies that

$$\left.\frac{\partial \mu(x)}{\partial \alpha}\right|_{\beta \text{ constant}} = \frac{\mu(x) - (1 - e^{-\mu(x)}) - \dfrac{1-F(x)}{1-F(r^*)}(\beta - (1 - e^{-\beta}))}{1 - \alpha + \alpha e^{-\mu(x)}} \leqslant 0.$$

By (A4), with (A17) satisfied:

$$\left.\frac{d\mu(x)}{d\alpha}\right|_{r^* \text{ constant}} = \frac{\partial \mu(x)}{\partial \alpha} + \frac{\partial \mu(x)}{\partial \beta}\frac{\partial \beta}{\partial \alpha} < 0.$$

Using (A13) and (A16),

$$0 = \nu_{r^*}\, dr^* + \delta \int_{r^*}^{1} \gamma'(\mu(x)) \left.\frac{d\mu(x)}{d\alpha}\right|_{r^* \text{ constant}} dx\, d\alpha,$$

which implies $dr^*/d\alpha < 0$. Note that ν_{r^*} accounts for the effect of changing r^* on μ.

V.3—*Derivatives with respect to* δ: $0 = \Psi_\beta\, d\beta + \Psi_{r^*}\, dr^*$. Thus $d\beta/d\delta$ and $dr^*/d\delta$ have opposite signs. We may rewrite ν to obtain

$$\begin{aligned}
\nu &= r^* - \delta\left(1 - \int_{r^*}^{1} \gamma(\mu(x))\, dx\right) \\
&= r^* - \delta\left(r^* + \int_{r^*}^{1} 1 - \gamma(\mu(x))\, dx\right) \\
&= (1-\delta) r^* - \delta(1-\delta) \int_{r^*}^{1} \frac{e^{\mu(x)} - \mu(x) - 1}{(1-\delta)e^{\mu(x)} + \delta}\, dx.
\end{aligned}$$

Thus, dividing by $(1-\delta)$,

$$\text{(A19)} \qquad 0 = r^* - \delta \int_{r^*}^{1} \frac{e^{\mu(x)} - \mu(x) - 1}{(1-\delta)e^{\mu(x)} + \delta}\, dx.$$

The right-hand side of (A19) is increasing in r^* because ν is increasing by (A16), and is decreasing in δ (note that by (A17), $\mu(x)$ is invariant to δ), so we have $dr^*/d\delta > 0$ and $d\beta/d\delta < 0$.

V.4—*The special case* $\alpha = \delta$ *and* $F(x) = x$: From $\alpha = \delta$ and (38)–(40),

$$\text{(A20)} \qquad (1-\alpha)\mu(x) + \alpha(1 - e^{-\mu(x)}) = b(1 - F(x))(1 - \alpha e^{-\beta}),$$

which yields

$$\text{(A21)} \qquad \mu'(x) = \frac{-bf(x)(1-\alpha e^{-\beta})}{1 - \alpha + \alpha e^{-\mu(x)}} = \frac{-b(1-\alpha e^{-\beta})}{1-\alpha+\alpha e^{-\mu(x)}}.$$

Therefore, by (A19),

$$\begin{aligned}
0 &= r^* - \int_{r^*}^{1} \frac{1 - e^{-\mu(x)}(1 + \mu(x))}{1 - \alpha + \alpha e^{-\mu(x)}}\, dx \\
&= r^* + \frac{\alpha}{b(1-\alpha e^{-\beta})} \int_{r^*}^{1} \left[1 - e^{-\mu(x)}(1+\mu(x))\right] \mu'(x)\, dx.
\end{aligned}$$

Multiplying by $b(1-\alpha e^{-\beta})$, and noting that $d(2+\mu)e^{-\mu}/dx = -e^{-\mu}(1+\mu)\mu'$, we have:

$$\text{(A22)} \qquad 0 = b(1-\alpha e^{-\beta})r^* + \alpha\left[\mu(x) + e^{-\mu(x)}(2+\mu(x))\right]\Big|_{r^*}^{1}$$
$$= b(1-\alpha e^{-\beta})r^* + \alpha\left[2-\beta-e^{-\beta}(2+\beta)\right].$$

From (39) for $x = r^*$,

$$(1-\alpha)\beta + \alpha(1-e^{-\beta}) - b(1-r^*)(1-\alpha e^{-\beta}) = 0,$$

or, equivalently,

$$\text{(A23)} \qquad b(1-\alpha e^{-\beta})r^* = b(1-\alpha e^{-\beta}) - (1-\alpha)\beta - \alpha(1-e^{-\beta}).$$

Putting (A22) and (A23) together, we obtain

$$s(\beta) \equiv b(1-\alpha e^{-\beta}) - \beta + \alpha\left[1-e^{-\beta}(1+\beta)\right] = 0, \quad \text{and}$$

$$\text{(A24)} \qquad r^* = 1 - \frac{(1-\alpha)\beta + \alpha(1-e^{-\beta})}{b(1-\alpha e^{-\beta})} = 1 - \frac{1}{b} - \frac{(1-\alpha)(\beta-1)}{b(1-\alpha e^{-\beta})}.$$

At $\alpha = 0$, $s(\beta) = b - \beta$, and therefore $\beta = b$ and this gives $r^* = 0$.

$$\lim_{\alpha\to 0}\frac{d\beta}{d\alpha} = -\frac{-be^{-\beta} - e^{-\beta}(1+\beta) + 1}{b\alpha e^{-\beta} - 1 + \beta\alpha e^{-\beta}} \to 1 - e^{-b}(1+2b).$$

This gives

$$\lim_{\alpha\to 0}\Phi = \lim_{\alpha\to 0}\frac{r^*}{\alpha} = \lim_{\alpha\to 0}\frac{\partial r^*}{\partial\alpha}$$
$$= \lim_{\alpha\to 0}\frac{\beta-1}{b(1-\alpha e^{-\beta})} - \frac{(1-\alpha)(\beta-1)e^{-\beta}}{b(1-\alpha e^{-\beta})^2} - \frac{(1-\alpha)\frac{\partial\beta}{\partial\alpha}}{b(1-\alpha e^{-\beta})} + \frac{(1-\alpha)(\beta-1)\alpha e^{-\beta}\frac{\partial\beta}{\partial\alpha}}{b(1-\alpha e^{-\beta})^2}$$
$$= \frac{b-1}{b} - \frac{(b-1)e^{-b}}{b} - \frac{1}{b}\left(1-e^{-b}(1+2b)\right)$$
$$= b^{-1}\left[b-2+e^{-b}(b+2)\right].$$

For the case $\alpha \to 1$, we consider two cases. This case requires some care because, at $\alpha = \delta = 1$, $s(\beta) = 0$ has multiple solutions if $b > 1$.

Case I—$\alpha = \delta \to 1$ *and* $b \leqslant 1$: $s(0) = 0$, and, as $\alpha \to 1$,

$$s'(\beta) = -\left(1-\alpha e^{-\beta}(b+\beta)\right) \leqslant -\left(1-\alpha e^{-\beta}(1+\beta)\right) \to -\left(1-e^{-\beta}(1+\beta)\right) \leqslant 0.$$

Thus, as $\alpha \to 1$, $\beta \to 0$. Moreover,

$$\lim_{\alpha\to 1}\frac{d\beta}{d\alpha} = \lim_{\alpha\to 1} -\frac{-be^{-\beta} - e^{-\beta}(1+\beta) + 1}{b\alpha e^{-\beta} - 1 + \beta\alpha e^{-\beta}} = \frac{b}{b-1}.$$

This gives

$$\lim_{\alpha\to 1} r^* = 1 - \frac{1}{b} - \frac{1}{b}\lim_{\alpha\to 1}\frac{(1-\alpha)(\beta-1)}{1-\alpha e^{-\beta}}$$
$$= 1 - \frac{1}{b} - \frac{1}{b}\lim_{\alpha\to 1}\frac{-(\beta-1) + (1-\alpha)\frac{d\beta}{d\alpha}}{-e^{-\beta} + \alpha e^{-\beta}\frac{d\beta}{d\alpha}}$$
$$= 1 - \frac{1}{b} - \frac{1}{b}\frac{1}{-1+\frac{b}{b-1}} = 0.$$

Case II—$\alpha = \delta \to 1$ *and* $b > 1$: Note that $s(\beta)$ takes a maximum at a point β_o satisfying $0 = \alpha e^{-\beta_o}(b + \beta_o) - 1$, and β_o is increasing in α. Thus the solution to $s(\beta) = 0$ exceeds β_o, and is therefore the positive solution to

$$0 = b(1 - e^{-\beta}) - \beta - e^{-\beta}(1 + \beta) + 1.$$

Consequently, by (A24), $r^* = 1 - (1/b)$.

V.5—*The Socially Efficient Reserve Price*: In computing the socially efficient reserve, it is no longer necessary that the reserve price r equal the discounted value of being a seller. Thus, we have from (14):

$$\Phi = \delta\Phi + \int_r^1 [(x - \delta\Phi - \delta\pi(x))e^{-\mu(x)} - (1-\delta)\pi(x)](-\mu'(x))\, dx$$

$$= \delta\Phi + (x - \delta\Phi - \delta\pi(x))e^{-\mu(x)}\big|_r^1 - \int_r^1 (1 - \delta\pi'(x))e^{-\mu(x)}\, dx$$

$$+ (1-\delta)\pi(x)\mu(x)\big|_r^1 - (1-\delta)\int_r^1 \pi'(x)\mu(x)\, dx$$

$$= \delta d\Phi + 1 - \delta\Phi - \delta\pi(1) - (r - \delta\Phi)e^{-\beta} - \int_r^1 (1-\delta)\pi'(x)\, dx$$

$$- (1-\delta)\int_r^1 \pi'(x)\mu(x)\, dx$$

$$= 1 - (r - \delta\Phi)e^{-\beta} - \int_r^1 \pi'(x)[1 + (1-\delta)\mu(x)]\, dx.$$

This yields

$$\Phi = \frac{1 - re^{-\beta} - \int_r^1 \pi'(x)[1 + (1-\delta)\mu(x)\, dx]}{1 - \delta e^{-\beta}}.$$

The present value of being an entering buyer is

$$\pi_o = \int_r^1 \pi(x) f(x)\, dx = \int_r^1 \pi'(x)(1 - F(x))\, dx,$$

where

$$\pi'(x) = \frac{e^{-\mu(x)}}{1 - \delta + \delta e^{-\mu(x)}} = \frac{1}{(1-\delta)e^{\mu(x)} + \delta}.$$

Thus the present value of an entering cohort is

$$V(r) = \Phi + b\pi_o$$

$$= (1 - \delta e^{-\beta})^{-1}\Big(1 - re^{-\beta} + \int_r^1 \pi'(x)\big[b(1 - F(x))(1 - \beta e^{-\beta})$$

$$- (1 + (1-\delta)\mu(x))\big]\, dx\Big).$$

We need the following fact:

$$\text{(A25)}\qquad 1 + \frac{b(1 - \delta e^{-\beta})(1 - F(x)) - (1 + (1-\delta)\mu(x))}{1 - \delta + \delta e^{-\mu(x)}} = \frac{(\alpha - \delta)[b(1 - F(x)) - \mu(x)]}{\alpha(1 - \delta + \delta e^{-\mu(x)})}.$$

To derive (A25), note that (A20) holds, because the derivation of (37)–(39) did not require $r^* = \delta\Phi$, and thus

$$\begin{aligned}
&1-\delta+\delta e^{-\mu(x)}+b(1-\delta e^{-\beta})(1-F(x))-1-(1-\delta)\mu(x)\\
&=b(1-\alpha e^{-\beta})(1-F(x))+(\alpha-\delta)be^{-\beta}(1-F(x))\\
&\quad+(1-\delta)+\delta e^{-\mu(x)}-1-(1-\delta)\mu(x)\\
&=(1-\alpha)\mu(x)+\alpha(1-e^{-\mu(x)})+(\alpha-\delta)be^{-\beta}(1-F(x))\\
&\quad+1-\delta+\delta e^{-\mu(x)}-1-(1-\delta)\mu(x)\\
&=(\alpha-\delta)\left[be^{-\beta}(1-F(x))+1-e^{-\mu(x)}-\mu(x)\right]\\
&=\frac{\alpha-\delta}{\alpha}\left[b(1-F(x))-\mu(x)\right].
\end{aligned}$$

(A20) is used to reach both the third and last lines of the derivation of (A25). Moreover, if $\beta=1$, again using (A20),

$$\text{(A26)}\qquad \begin{aligned}
b(1-F(x))&=\frac{1-\alpha}{1-\alpha e^{-\beta}}\mu(x)+\frac{\alpha}{1-\alpha e^{-\beta}}(1-e^{-\mu(x)})\\
&>\frac{1-\alpha}{1-\alpha e^{-\beta}}\mu(x)+\frac{\alpha(1-e^{-\beta})}{1-\alpha e^{-\beta}}\\
&>\frac{1-\alpha}{1-\alpha e^{-\beta}}\mu(x)+\left(1-\frac{1-\alpha}{1-\alpha e^{-\beta}}\right)1\\
&>\mu(x),\quad\text{since}\quad \mu(x)\leqslant\beta=1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V'(r)&=\frac{1}{1-\delta e^{-\beta}}\Bigg(-V(r)\delta e^{-\beta}\frac{\partial\beta}{\partial r}+re^{-\beta}\frac{\partial\beta}{\partial r}-e^{-\beta}\\
&\qquad-\pi'(r)\left[b(1-F(r))(1-\delta e^{-\beta})-(1+(1-\delta)\beta)\right]\\
&\qquad+\int_r^1\pi'(x)b(1-F(x))\delta e^{-\beta}\frac{\partial\beta}{\partial r}\,dx-\int_r^1\pi'(x)(1-\delta)\frac{\partial\mu(x)}{\partial r}\,dx\\
&\qquad-(1-\delta)\int_r^1\frac{e^{\mu(x)}}{\left((1-\delta)e^{\mu(x)}+\delta\right)^2}\left[b(1-F(x))(1-\delta e^{-\beta})\right.\\
&\qquad\qquad\left.-(1+(1-\delta)\mu(x))\right]\frac{\partial\mu(x)}{\partial r}\,dx\Bigg)\\
&=(1-\delta e^{-\beta})^{-1}\Bigg[\frac{\partial\beta}{\partial r}e^{-\beta}(-\delta V(r)+r+\delta b\pi_o)\\
&\qquad-\left(e^{-\beta}+\frac{b(1-F(r))(1-\delta e^{-\beta})-(1+(1-\delta)\beta)}{(1-\delta)e^{\beta}+\delta}\right)\\
&\qquad-(1-\delta)\int_r^1\left[(1-\delta)e^{\mu(x)}+\delta\right]^{-1}\frac{\partial\mu(x)}{\partial r}\\
&\qquad\times\left(1+\frac{b(1-F(x))(1-\delta e^{-\beta})-(1+(1-\delta)\mu(x))}{(1-\delta)+\delta e^{-\mu(x)}}\right)dx\Bigg]\\
&=\Bigg[\frac{\partial\beta}{\partial r}e^{-\beta}(r-\delta\Phi)-\frac{e^{-\beta}(\alpha-\delta)\left[b(1-F(r))-\beta\right]}{\alpha(1-\delta+\delta e^{-\beta})}\\
&\qquad-\frac{(1-\delta)(\alpha-\delta)}{\alpha}\int_r^1\frac{\partial\mu(x)}{\partial r}\,\frac{e^{-\mu(x)}}{1-\delta+\delta e^{-\mu(x)}}\,\frac{b(1-F(x))-\mu(x)}{1-\delta+\delta e^{-\mu(x)}}\,dx\Bigg].
\end{aligned}$$

Now, if $\alpha = \delta$, the second and third terms vanish, and $V'(r) \gtreqless 0$ as $r \lesseqgtr \delta\Phi$, by (A6), so the efficient reserve price is also the equilibrium. If $b(1 - F(r)) = 1$ at the point where $r^* = \delta\Phi$, then $\beta = 1$ and the first two terms vanish. By (A26) and (A7), we have $\operatorname{sgn} V'(r^*) = \operatorname{sgn}(\alpha - \delta) > 0$. Thus r^* is inefficiently low.

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