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How to Set Minimum Acceptable Bids, with an Application to Real Estate Auctions

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Abstract: In a general auction model with affiliated signals, common components to valuations and endogenous entry, we compute the equilibrium bidding strategies and outcomes, and derive a lower bound on the optimal reserve price. This lower bound can be computed using data on past auctions combined with information about the subsequent sales prices of unsold goods. We illustrate how to compute the lower bound using data from real estate auctions.

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There are many situations in which a seller, often a government, auctions many similar items over a long period of time. For example, over the past several decades, the Federal Deposit Insurance Corporation (FDIC) and the Resolution Trust Corporation (RTC) have auctioned tens of thousands of houses for tens of billions of dollars. Over the past thirty years, the U.S. Department of the Interior has auctioned billions of dollars worth of timber cutting rights and off-shore oil leases. Sales of treasury bills are in the trillions of dollars. This paper provides a procedure for increasing the seller's revenue over that obtained by *ad hoc* formulae used in practice by using historical data to improve on the minimum acceptable bid, or reserve price, imposed in the auction. Because the procedure is applicable to environments of considerable economic value, including not only real estate but also oil and other mineral rights, timber, radio spectrum and treasury bills, there is a potential for application of our theory to create a significant amount of increased revenue.

In contrast to much of the literature, we study an environment which allows for affiliation in the signals and common components to value, and in which participation is endogenously determined. With some important exceptions, the empirical auction literature has concentrated on the *independent private values* environment in which bidders know their own valuations and these valuations are independently distributed.[1] Such models cannot account for either correlation in valuations, as would occur if there are common factors that influence value and vary from auction to auction, or in unobserved factors affecting valuations that are common to the bidders.[2] These factors are clearly important in any real world auction environment, as Paul

[1] The most important theoretical treatment is Paul Milgrom and Robert Weber (1982), which developed the mathematical tools used in the present study. The auction literature is surveyed in McAfee and John McMillan (1987a). More specialized surveys are provided by Milgrom (1988) and Robert Wilson (1991). Optimal auctions with correlated values were studied by Jacques Cremer and Richard McLean (1985), McAfee, McMillan, and Philip Reny (1989), and McAfee and Reny (1991).

[2] Even if bidders know their own value for the item being sold, it would be rather surprising if these values weren't correlated through unobserved factors. For example, the desirability of a work of art purchased purely for private viewing is likely to be correlated across bidders. More generally, bidders only receive an estimate of the value, and the realized value will depend on unobserved factors correlated with all of the bidders' signals; e.g. the amount of oil in a tract is unobserved prior to drilling, but is presumably correlated with all the bidders' signals. In addition, the potential for resale at an uncertain future price induces correlation in the bidders' valuations.

Milgrom and Robert Weber (1982) persuasively argue. In addition, the auction literature has focused on the case of an exogenous set of bidders. In many real situations, bidders are attracted to the auction by potential profits, and changes in the selling mechanism will change the bidders' participation decisions.

In all but the simplest environments, optimal selling mechanisms tend to be very complicated and depend on the distributions of signals, utility functions and other aspects of the environment that are not actually observable but assumed known in order to fully specify a model. While we will also assume that agents in the model know the distributions of signals and utility functions of the other agents, in contrast to the existing literature, we simply derive a lower bound on the optimal reserve price that does not depend on specific knowledge of the distributions or utility functions posited in the model.[3] That is, the lower bound will be distribution-free.[4]

This approach is an extension of the analysis in McAfee and Vincent (1992), which applied a related analysis to the case of off-shore oil auctions. In this study we allow for more general valuation functions, (the earlier study restricted attention to a pure common value environment) and apply the analysis to a broader class of auction mechanisms including first price, second price and oral auctions. Most significantly, the approach offered in this paper does require the observation of the *ex post* value of the object. In most cases, observations of the *ex post* value of sold objects will be impossible; the OCS oil lease auction data, studied by Kenneth Hendricks and Robert Porter, with coauthors (1987, 1990, 1992), is an exception in this regard.

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[3] A differentiated feature of the model in this paper is that it incorporates entry decisions by bidders. In addition to our earlier study, McAfee and McMillan (1987b,c), Harstad (1990) and Levin and Smith (1994) examine endogenous and stochastic participation in auctions.

[4] Distributions and utility functions are the primitives of auction theory and we follow the literature in assuming that these primitives are common knowledge of the bidders. Our constructed lower bound is observable in many auction data sets. In contrast, setting an optimal reserve price in the relatively well-behaved independent private values auction requires knowledge of the distribution of valuations.

Furthermore, the present study computes improvements for potentially large adjustments to the reserve, while the previous study applied only to small changes. We consider our approach to be more robust than the optimal auctions approach because it depends on fewer assumptions and less knowledge on the part of the seller. In addition, by focusing on a simple improvement that a seller might reasonably adopt rather than a complex optimal auction, our approach is more practical.

Consider a sequence of similar items sold by auction. These items could be houses, off-shore oil rights, or other related items. We consider how to use data from early auctions to adjust the reserve price for the later sales. We presume that items that fail to sell have realized values prior to the subsequent auctions of new items. For example, items that fail to sell in early auctions are likely to be sold eventually. In particular, in the real estate sales application, houses that failed to sell in early auctions were sold later by bargaining or subsequent auctions, and a price for the seller was realized. This later realized price, discounted to the time of the initial sale attempt, is used to determine whether expected revenues will increase if a higher reserve price is imposed in subsequent auction of new but similar items.

We show that the discounted expected sale price of items that failed to sell in past auctions is a lower bound for the optimal reserve, if this average sale price exceeds the past reserve. It is useful to distinguish *ex ante* considerations of the seller, which occur prior to the participation decisions, from *ex post* considerations, which occur at the time of bidding. Endogenous participation implies that the bidders earn zero *ex ante* expected rents. Thus, on average, the entire gains from trade accrue to the seller, and in contrast to models with an exogenous number of bidders, the seller wishes to post an *ex ante* efficient reserve price. However, efficiency *ex post* means setting a reserve price equal to the seller's value associated with retaining the object. For a large class of environments, the *ex ante* efficient reserve exceeds the *ex post* efficient reserve, because of an entry externality. Therefore, the seller *ex ante* should post a reserve price above the seller's opportunity cost of sale.

Two complications arise in calculating the value of the object to the seller. First, the *ex post* efficient reserve holds the participation strategy of the bidders constant and equates the value of items that just fail to sell at the current reserve to the seller's expected value of these items[5] rather than the average value of all unsold items. Realistically, though, it is the average value that is typically observable. Second, even the average value of unsold items will tend to vary with the reserve. If unsold goods are kept by the seller and used in some alternative capacity of known value, then the seller could simply observe the value of unsold goods. However, more plausibly, the value of the item depends on some imperfectly observed intrinsic quality. This is particularly the case when the opportunity cost of sale in the present auction is the value of sale in a subsequent auction. This quality will tend to be correlated with buyers' willingness to pay, and thus changing the reserve price will change the quality, and hence the expected value to the seller, of unsold items. Therefore, changes in the reserve price change the composition of the set of objects that fail to sell, a classic case of sample selection bias.

The sample selection problem implies that the present discounted expected value of unsold items is a lower bound on the appropriate reserve price. Suppose that the current reserve price is less than the average resale value of objects that fail to sell in the current auction. Then the value of marginal objects that fail to sell at the current reserve exceeds the average value of objects that fail to sell, which by assumption exceeded the reserve. Thus, whenever the average present value of future resale exceeds the reserve price, the reserve price should be raised to *at least* this average value. Raising the reserve will, of course, increase the value of the marginal

[5] That is, the ex post efficient reserve, r, must satisfy the condition that it equal the seller's value of items that fail to sell at a reserve r but would sell at any reserve r-ε for small $\varepsilon>0$, since this equates the seller's value of selling and not selling at the margin. We will refer to this value as the value of marginal items.

good that fails to sell. Thus, the approach offers a conservative but specific estimate of how much the reserve can be increased since raising the reserve to the average value will result in a reserve that is still too low. This is in contrast to the approach in McAfee and Vincent (1992) which only yields a statistic that determines if *some* increase in reserve will raise expected revenues. As an example of how the approach may be used, in Section 3 we offer an illustrative case of private real estate auctions. The data set we were able to acquire is too small to draw any truly persuasive conclusions but it shows that, given plausibly available data, the technique is implementable.

1. The Affiliated Values Model with Endogenous Entry: Bidder Behavior

We assume that there is a large number, n, of potential bidders, sufficiently large so that even without a posted reserve, it is not an equilibrium for all bidders to bid. For a cost s, each bidder i can obtain a signal x_i which is a realization of the random variable X_i with cumulative distribution function $F(X_i|\theta)$, where θ is a vector of variables not observed by any agent. Bidders who do not pay s are assumed not to bid, perhaps because they do not learn about the existence of the auction without paying s. We call θ the common component. In applications, θ represents all aspects of the item for sale that affect the value of the item but are not observed by the agents. By convention, higher values of θ correspond to higher values of the good. Bidders' signals are independently distributed, conditional on θ. We also assume that X_i has a smooth density $f(X_i|\theta)$. The value of the good to the buyer, given realized signal x and common component θ, is $u(x,\theta)$. The payoff u is assumed to be nondecreasing in all of its arguments.

This model is less general than Milgrom and Weber's (1982) model in two respects. First, Milgrom and Weber do not assume the signals are conditionally independent. Second, other buyers' signals do not enter into the payoff u of a given buyer i. We assume, following

Milgrom and Weber, that each of the random variables, $X_1,\ldots,X_n$ are affiliated with the common component, θ.[6] As is standard, u, f, s, n and the distribution of θ are common knowledge among the potential buyers.

The model is usefully illustrated by considering the sale of a house. The variables θ represent all of the unobservable attributes of the house, measured so that higher values of θ represent higher quality. Potential buyers decide whether to examine the property; those that conduct an examination incur a cost s. Each buyer forms an estimate of the value of the property, denoted x, which is a sufficient statistic for everything observable about the house, from the color of the appliances to the sagging roof.[7] Armed with the estimate x, buyers submit bids in an auction. The seller's value of the house if the house fails to sell is denoted σ.

The seller holds an auction with reserve price r. The auction form may be any of a first or second price auction or oral ascending bid auction. In such auctions bidders will not participate unless their signal is sufficiently high, at a level Milgrom and Weber (1982) call the screening level, which we denote by x_r. The screening level is the signal such that, knowing that all other bidders either didn't participate or observed signals less that x_r (and hence didn't submit bids), a buyer with signal equal to x_r just breaks even by paying r for the good.

The timing is as follows. First, the seller announces r. Second, the buyers choose whether or not to pay a cost s to acquire a signal. Only the symmetric random participation equilibrium will be considered, in which buyers choose to acquire a signal with probability

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[6] For two random variables, affiliation is also known as the Monotone Likelihood Ratio Property. For general functions as well as densities, affiliation is called log supermodularity. A twice differentiable function f is supermodular if the cross-partials are non-negative. f is log supermodular if $\log(f)$ is supermodular. See Milgrom and Roberts (1990) for an exhaustive set of consequences of supermodularity. Affiliation may be thought of as a strong form of local positive correlation - that is, two random variables are affiliated if and only if increasing functions of these random variables are positively correlated, on every sublattice of the variables' domain. One consequence of affiliation, used repeatedly in the present analysis, is that $\frac{1-F(x|\theta)}{f(x|\theta)}$ is increasing in θ.

[7] It is a restriction that x be univariate. To our knowledge, there is no theory of bidding with multidimensional signals that does not readily reduce to the univariate signal case.

$\rho \in (0,1)$.[8] Third, informed buyers submit bids; bids less than r are ignored. Fourth, the bidder with the highest (final) submitted bid in excess of r obtains the object, and pays a price that will depend on the specific auction form employed. If no buyer submits a bid exceeding r, the seller keeps the item and obtains the value $\sigma(\theta)$. The seller's value σ is assumed nondecreasing in θ. We consider in the theory the sale of a single object, and leave implicit in σ the means by which the seller realizes the opportunity cost of sale. Bidders who don't purchase a signal obtain zero. Bidders who purchase a signal but fail to obtain the object obtain the von Neumann-Morgenstern utility $-s$, while bidders who pay p for the object obtain $u(x_i,\theta)-p-s$.

The variable, $\sigma(\theta)$, represents the opportunity cost to the seller of selling the object in the current auction. Since θ will be assumed to be (initially) unobservable to the seller, this opportunity cost is perceived as a random variable at the time the auction takes place. The implementation of our approach, however, requires that an estimate of $\sigma(\theta)$ be available eventually. This value may become known to the seller through use of the object and the information could be the source of the data. More likely, however, $\sigma(\theta)$ represents the revenues the seller can obtain through the sale of the object at some other institution at some other time. In applications, this data on later sales is required (and is often available). Note that, in this case, the assumption that the payoff of a bidder who buys a signal but fails to obtain the object at auction is $-s$ also requires that this bidder not participate in the subsequent sale event. In the case of real estate auctions, bidders who have time specific needs for property (they need the house now and not six months from now) would fall in this category. In other auctions, such as antique auctions, auction houses themselves move unsold product to different geographical sites and the bidding

[8] Asymmetric equilibria, with some buyers participating with certainty and others not at all, exist. These equilibria lead to qualitatively similar results, and indeed avoid some of the problems associated with randomized participation. However, they also introduce an "integer problem," in that participation tends to be a step function of the reserve. See McAfee and McMillan (1987c) for an analysis of such equilibria in the independent private values framework.

audience may well differ.[9] Of course, the extent to which this assumption is valid should be monitored in each specific application.

A standard approach in analyzing equilibria in auctions with a fixed number of bidders is to conjecture that bids are monotonic functions of signals. This conjecture is then used to determine a probability of winning the auction for a given bid, *b,* and to determine the bidder's expected utility at the bid, *b.* The best response bid is calculated, symmetry imposed and the resulting bid function is then checked for the monotonicity assumption. In affiliated values auctions, monotonicity is generally implied by the supermodularity assumption embodied in affiliation. For a fixed participation ratio, ρ, we can conduct a similar analysis. Let $B(\bullet;\rho)$ denote an equilibrium bidding function.[10] A sufficient condition for $B(\bullet;\rho)$ to be nondecreasing is the log supermodularity of $1-\rho(1-F(x|\theta))$,[11] or

$$(1) \qquad (\forall x \geq x_r, \forall \theta' \geq \theta,) \quad \frac{f(x|\theta')}{1-\rho(1-F(x|\theta'))} \geq \frac{f(x|\theta)}{1-\rho(1-F(x|\theta))}.$$

Condition (1) is a sufficient condition for all the intuitive monotonicities derived below, and so we assume it here, although we note below when it is used. Affiliation of f implies (1) for $\rho=1$.[12] The meaning of assumption (1) is illustrated in the following thought experiment.

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[9] We are not sure how our results would be affected if failed bidders also participate in subsequent attempts to resell the object. The additional dynamic incentives render the model very complex. McAfee and Vincent (1997) illustrate optimal reserve price policies in such environments. If the auctioneer can commit to keeping the object off the market for a long enough period of time, then much of our analysis would remain essentially valid. If this commitment ability is absent, though, current bidding behavior will be affected by the opportunity to acquire the object later.

[10] In sealed bid auctions, $B(\bullet;\rho)$ is a function of a bidder's signal alone. In ascending bid auctions, it is also a function of the bids at which rival bidders drop out. In this latter case, monotonicity means $B(\bullet;\rho)$ is increasing in the signal x for all values of drop out bids of rivals.

[11] See Athey (1995) for a discussion of log supermodularity and its application. The proof that this condition is sufficient for monotonicity is an adaptation of proofs in Milgrom and Weber (1982).

[12] Since Milgrom and Weber (1982) have $\rho=1$, (1) holds in their model by affiliation. Inequality (1) must fail to hold globally if ρ is very close to zero, and in particular fails for x near its lower bound, as $\rho\rightarrow 0$. However, we need (1) only for $x \geq x_r$; this is feasible even for $\rho=0$. While a somewhat weaker condition will suffice for monotonicity of the bidding function (in particular, the log supermodularity of $(1-\rho(1-F))^{n-2} f^2$ would suffice), (1) is nevertheless the "natural" sufficient condition to combine with affiliation, especially as (1) is independent of n.

Consider first the event of receiving exactly one bid, $B(x_r;\rho)$, and second, observing no bids at all. Assumption (1) implies that the expected value of the good given the first event exceeds the expected value of the good given the second event (this is proved in Lemma 5 below). There are two circumstances under which a buyer does not bid: either the buyer received a signal less than x_r, or the buyer did not obtain a signal at all. That a buyer obtained no signal is "good news" (Milgrom (1981)) about the value of the object, relative to the knowledge that the buyer's signal was very low. Assumption (1) implies that it is better news to see a signal exactly equal to x_r, and hence a marginal bid, than to see no bid at all. Whether assumption (1) is plausible, then, depends on whether x_r is sufficiently large that the signal x_r is good news. The value of x_r depends on r.

We denote expectation over θ by E_θ. We denote expected equilibrium profits of a bidder with signal, x, by $\pi(x)$. Note that in environments other than independent private values, this function will typically differ depending on the auction form that is used. Nevertheless, our results are robust to this indeterminacy. The screening level satisfies $\pi(x_r)=0$, or

(2) $$0 = E_\theta\left[(u(x_r,\theta) - r)(1 - \rho(1 - F(x_r|\theta)))^{n-1} f(x_r|\theta)\right].$$

The participation decision, which determines ρ, is given by bidders' indifference between expending s to become informed, and obtaining zero. This implies

(3) $$s = E_\theta \int_{x_r}^{\infty} \pi(x) f(x|\theta)\, dx.$$

Equations (2) and (3) jointly determine x_r and ρ.

One naturally expects that an increase in the reserve price r would increase the screening level x_r and decrease the participation probability ρ. That is,

(4) $$\frac{dx_r}{dr} > 0, \text{ and } \frac{d\rho}{dr} < 0.$$

However, this “natural” comparative statics does not hold in all environments. Indeed, it is possible to show that ρ does not necessarily fall monotonically as r rises. Consider a common value model as follows. Let $u(x,\theta)=\theta$, $\theta\in\{0,1\}$, $Prob[\theta=0]=.5, F(x|\theta)=x^{\theta+1}$, and consider a single object sold at a second price auction. Figure 1 shows how x_r and ρ change with r for the case with the maximum number of bidders equal to five. Although the non-monotonicity in ρ is slight it appears robust and is more easily generated with higher values of n than low values.

One reason for the non-monotonicity lies in the peculiar effect that increasing the number of bidders may have on expected bids and expected seller revenues in the presence of common values. Steven Matthews (1984) shows that expected buyer profits need not be monotonic in participation. For example, if a rise in r leads to an increase in x_r, holding ρ fixed, the initial impact may be to lower bidder profits. In order to continue to satisfy the zero profit entry condition, it may be necessary either to raise or lower the expected number of bidders by raising or lowering ρ depending on the effect of the number of bidders on bidder profits in the particular environment.

This observation suggests that when there is no ambiguity concerning the effect of increasing the number of bidders on expected revenue, then the ambiguity of the impact of r on x_r, and ρ also disappears. With second price auctions, under private values, even with affiliation, this is indeed the case, as the following lemma shows. Lemma 1 does not require assumption (1).[13]

Lemma 1: *Suppose* $u(x,\theta)=x$, *Then (4) holds in second price auctions.*

All proofs are contained within the appendix.

It is readily shown by differentiating (2) that at least one of the inequalities in (4) must hold. In addition, locally around ρ=0, (4) holds, as we demonstrate below for second price auctions. This result depends on (1) holding. As ρ→0, inequality (1) requires that x_r be

[13] With private value second price auctions, equilibrium bid functions are independent of the number of bidders. It is not known if a similar result to Lemma 1 can be shown for the case of first price auctions. Pinkse and Tan (2000) have shown that expected revenues can fall as the number of bidders increases in affiliated private values first price auctions, so the intuition suggests that we cannot always be assured that (4) holds in this case.

sufficiently large. For example, suppose θ has support $[\theta_L, \theta_H]$. For $F(x|\theta) = x^{\theta+1}$, (1) holds if $x_r \geq e^{-1/(\theta_H+1)}$. Similarly, if $F(x|\theta) = 1 - e^{-\lambda x/\theta}$, (1) is equivalent to $x_r \geq \theta_H/\lambda$.

Lemma 2: *For s sufficiently large, so that ρ is close to 0, (4) holds in second price auctions.*

There is a possibility of multiple solutions to (2) and (3), because expected buyer profits need not be monotonic in participation. We ignore this complication in the remainder of the analysis. Stability requires that, as participation increases, then expected profits fall, for otherwise a slight increase in bidders' beliefs about participation would lead to increased participation, reinforcing the expectation. Given stability,[14] in the appendix, there is a simple to state, but difficult to interpret, sufficient condition imposed on the distribution F for x_r to rise and ρ to fall with r.

2. The Effect of Reserve Prices on Seller Profits.

Since the *ex ante* surplus of buyers is zero, the seller obtains the gains from trade net of entry costs.[15] Thus the seller wishes to select an efficient auction. Intuitively, this requires that the seller sell only when the expected value of the object to a bidder exceeds the seller's value, denoted $\sigma(\theta)$. However, we assume that the seller does not know the realization of θ, and thus cannot trivially set an *ex ante* efficient reserve price.[16] Denote the seller's surplus by Ψ. Assuming that the bids are monotonic in bidder signals and exploiting the fact that, in equilibrium, ex ante bidder profits are zero, a useful expression for Ψ is

$$(5)\quad \Psi = E_\theta\left[\sigma(\theta) + \int_{x_r}^{\infty}(u(x,\theta) - \sigma(\theta))\, n(1 - \rho(1 - F(x|\theta)))^{n-1}\rho f(x|\theta)\, dx - n\rho s\right]$$

[14] Our formulation of stability depends not on (3) directly, but on (3) with r replaced with the value solved out from (2).

[15] A similar result is noted by McAfee and Vincent (1992) and Levin and Smith (1994).

[16] If the seller knows θ, Milgrom and Weber (1982) show that the seller should announce θ to the bidders, in an environment where participation is exogenous.

Thus, the seller's payoff is the value of not selling, $\sigma(\theta)$, plus the net gains from trade when trade occurs, u-σ, evaluated at the highest signal received, minus the cost of buyer participation, $n\rho s$.

Expressed as in (5), the seller's value depends on the reserve r only through the dependence of x_r and ρ on r. This fact explains why the analysis does not rely on the specific form of auction used. Note, however, the result does not imply that seller expected revenues are independent of the auction mechanism. The failure of revenue equivalence in affiliated auctions implies that different auction mechanisms will generate different values for x_r and ρ for a given reserve price, r. For example, hold x_r and r fixed, and consider the equilibrium value of ρ from a first price auction. Since we know that expected payments in second price auctions are weakly higher than in first price auctions, it must be the case that expected bidder profits would be lower at the same value of ρ. Since this value of ρ yielded zero profits including entry costs in the first price auction, the same values of x_r and ρ can not represent an equilibrium in a second price auction.

Equation (5) makes clear that the effect of the reserve price instrument for a seller's expected revenues depends on how it changes x_r and ρ. We have shown that these effects can be ambiguous. In this section, however, in this section we explore the consequences of changes in r, when x_r and ρ change with r in the expected ways. Lemma 3 characterizes the effects of changes in x_r and ρ on the seller's payoff, which is used in establishing the effect of a change in the reserve, using (4).

Lemma 3: *Assume (4) holds.*[17]

(6) $$\frac{\partial \Psi}{\partial x_r} = -E_\theta[(r - \sigma(\theta))n(1 - \rho(1 - F(x_r|\theta)))^{n-1}\rho f(x_r|\theta)],$$

(7) $$\frac{\partial \Psi}{\partial \rho} \leq E_\theta[(r - \sigma(\theta))n(1 - \rho(1 - F(x_r|\theta)))^{n-1}(1 - F(x_r|\theta))].$$

[17]The provided proof of Lemma 3 is for second price auctions. A similar proof holds for first price and oral auctions and is available from the authors on request.

Lemma 3 computes the value of increasing both the screening level x_r and the participation probability ρ to the seller, and in both cases relates these values to the difference between the reserve price and the seller's value. Increasing the screening value increases the seller's payoff if and only if the seller's value is less than the reserve price, evaluated at the circumstance where a buyer is just indifferent between paying the reserve and not purchasing (that is, the seller's expected value for the marginal property).

The reason that (7) holds with an inequality rather than an equality arises from linkage principle arguments. Consider the pure common values case, so u_x=0. In this case, adding an extra bidder increases the likelihood that the good sells, which provides an increase in gains from trade accounted for in (7). However, additional participation also increases the likelihood that there are two or more bidders, a socially wasteful duplication of entry costs. (This loss arising from duplication in entry costs is mitigated when bidders with higher signals have higher values.) Let $X_{(1)}$, $X_{(2)}$ represent the highest and second highest signals, and B the price paid. Then there is a social efficiency gain of $u(X_{(1)},\theta) - u(X_{(2)},\theta)$ when a buyer with a higher signal is added by increased participation (this effect is zero in the common value extreme case). However, part of the gain, $u(X_{(1)},\theta) - B$, is the winning bidders' profit which does not accrue to the seller but goes to pay the costs s of participation, and therefore should be subtracted from the gains from trade for a net gain of $B - u(X_{(2)},\theta)$. But the average value of this expression is negative in general. In second price private value auctions with or without affiliation, $B = u(X_{(2)},\theta)$ and the term vanishes. With some common value element, it is well-known that conditional on knowing the highest signal, $B \leq u(X_{(2)},\theta)$ and therefore the term is negative on average. In first price auctions, as long as there is affiliation the term is negative even in the pure private value case.[18]

[18]Does a second price auction with *ex post* efficient reserve attract too many bidders? The answer is yes. Suppose the reserve price is chosen in such a way that (6) is zero, which is the *ex post* efficient reserve price. Then the right hand side of (7) is nonpositive.

We are now in a position to characterize a lower bound on the optimal reserve price, based on historical data for auctions of similar items. Theorem 4 depends on both (1) and (4). Define $\tilde{E}_x$ to be the expectation over θ conditional on the highest signal being x.

Theorem 4: *Fix a reserve price* r_0*, and suppose that* $r_0 < \tilde{E}_{x_{r_0}}[\sigma(\theta)] = \sigma_0$*, that is, the expected value of properties that just fail to sell is greater than the reserve. Then* $\left.\frac{d\Psi}{dr}\right|_{r_0 \le r \le \sigma_0} \geq 0$. *Expected seller profits rise with an increase in the reserve up to* σ_0 .

Theorem 4 indicates that if the expected value to a seller, σ_0, of properties that just fail to sell at a reserve price, r_0, is greater than r_0, then seller expected profits are rising in the reserve price for any reserve between r_0 and σ_0. In Figure 2, we graphically illustrate Theorem 4. The curve represents the expected value of marginal unsold items, $\tilde{E}[\sigma(\theta)]$. This depends on the reserve price through its effect on x_r and ρ. If the reserve price is less than $\tilde{E}[\sigma(\theta)]$, increasing the reserve to $\tilde{E}[\sigma(\theta)]$ will still leave the reserve below the optimal one, denoted r^*. That $\tilde{E}[\sigma(\theta)]$ is increasing in r is a consequence of affiliation, the monotonicity of σ, and (4). However, the uniqueness illustrated in Figure 2 cannot be guaranteed without placing further restrictions on σ.

Theorem 4 implies the following. Consider sales of houses, and suppose that the reserve price is less than the present value of resale for houses right at the margin, i.e. those with a bidder just indifferent between bidding and not. Then it is profitable for the seller to raise the reserve price to the present value of resale for those houses.

By itself, the implication of Theorem 4 would be difficult to implement empirically, because it is difficult to establish which houses were at the margin, that is, which houses had a

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Consequently, if the reserve price is chosen in such a way that the seller's payoff is maximized with respect to the screening level, then the participation probability ρ is too high. This observation, which appears empirically useless, does not depend on either assumptions (1) or (4).

bidder indifferent to bidding on them.[19] However, the average value of unsold houses is less than the value of marginal unsold houses. While this proposition seems intuitive, it in fact relies upon inequality (1) for a proof. The reason the proposition might be less than obvious is that failing to attract any bidders at all may be a result of no bidders becoming informed, which could be good news about the value of the property, as compared with the event of attracting one marginal bidder. However, assumption (1) implies that attracting the marginal bidder is overall better news than the event of attracting no bidders at all, as the following lemma shows.

Lemma 5: $\dfrac{E_\theta[\sigma(\theta)(1-\rho(1-F(x_r|\theta)))^{n-1} f(x_r|\theta)]}{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1} f(x_r|\theta)]} \geq \dfrac{E_\theta[\sigma(\theta)(1-\rho(1-F(x_r|\theta)))^{n}]}{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n}]} \equiv \bar{\sigma}.$

Lemma 5 shows that the value of the good to the seller in the event that no bidders are attracted is less than the value of the good to the seller in the event that one marginal bid is attracted. Combining Theorem 4 and Lemma 5, we have:

Corollary 6: *Suppose that the average value* $\bar{\sigma}$ *of unsold items exceeds the reserve price. Then raising the reserve price to* $\bar{\sigma}$ *increases seller revenue.*

Corollary 6 depends only on observables, and contains a testable prediction. In particular, the average value *to the seller* of unsold items is often observable by the seller. In the data considered below, we observe houses that don't sell in an auction, and the later sale of these houses. From data on the later sale price, we construct a present value, and find that the present value to the seller of real estate that does not sell is about 93% of appraised value. This estimate is a lower bound of the appropriate reserve price.

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[19] McAfee and Vincent (1992) propose a methodology for solving this problem, for common value auctions. The strategy requires the observation of *ex post* valuations, such as are available for the OCS oil auctions studied by Hendricks, Porter and Boudreau (1987). The technique is to look at the properties that received bids close to the reserve price, and estimate the distribution of *ex post* valuations conditional on a marginal winning bid. The entire database is used to estimate the expected winning bid conditional on the *ex post* value. Given this distribution of values for properties receiving marginal bids, it is then possible to estimate the average winning bid of marginal properties, which, with appropriate discounting, is approximately what could be expected if the properties were re-auctioned later.

What about the reverse implication? There are two obstacles in attempting to apply the analysis to learn when reserve prices should be lowered. Recall Lemma 3. With some common value element, the inequality in (7) will be strict because of the linkage principle. Thus, Theorem 4 cannot be extended to learn when the reserve price should be lowered even if data was available to show that the seller's use value of objects that just fail to sell on the margin was below the reserve price. Second, even without this hurdle, Lemma 5 shows us that if all we know is the *average* value of unsold properties, then learning that the reserve price lies above this average value does not warrant concluding that the reserve price also lies above the value of marginally unsold properties. The analysis, therefore, offers only a one-directional test.

Corollary 6 and Theorem 4 both state quite intuitive economic propositions. Effectively, both results state that one shouldn't sell items for less than their value in an alternative use. These propositions hold in a broad set of circumstances. It is remarkable how difficult it is to establish what seem like obvious propositions. The source of the difficulty, of course, is the endogenous entry of bidders; alterations in the reserve price may have adverse impact on participation in auctions, and an intuition arising from models with exogenous participation doesn't account for this effect.

Typically, the seller who fails to sell in the current auction will generally attempt to sell again later; this is the case in the real estate auctions we present below. It is important to realize that our theory accommodates this case. The theory itself accounts for the sample selection bias, in that the distribution of θ for items that fail to sell explicitly depends on the reserve price.[20] Thus, we are considering the appropriate class of items that fail to sell. Furthermore, the theory

[20] Recall, as noted above, we require that the fact that unsold objects may be put up for sale at a later time does not alter bidding behavior in the initial auction. In the First Interstate Bank data on real estate auctions, the average time to resale is about 3 months. In other real estate auctions such as FDIC distressed property, the average time to resell is over a half a year. For bidders on properties who are time sensitive, this assumption will be valid.For an analysis of dynamic behavior in auctions see McAfee and Vincent (1997).

suggests a way to enhance revenue, and therefore suggests a means of increasing the value of items that fail to sell, that is, increasing σ. As the theory will suggest that the average value of σ conditional on no sale is a lower bound for the optimal reserve, the historical average value of σ remains a lower bound on the optimal reserve after steps are taken to increase σ.

Is it possible, following a failure to meet the reserve price in the first auction, for the average sale price in the second auction to exceed the reserve price? The answer is yes if there is enough of a private value component. Consider the following simple example. A seller has zero use value for a property and attempts to sell it in two auctions. The auctions are separated enough either in time or space so that a different group of bidders (both of size n) participate in each auction. Suppose the environment is independent private values with a support, say, of $[0,\infty)$. In the final auction, with an entry probability of $\rho<1$ when the reserve price is zero, the fact that the seller's expected revenue corresponds to social surplus implies that her optimal reserve price is zero.As long as s is not too large, the expected sale price, p, will be strictly positive. The discounted value of this price serves as σ in the first auction (the IPV assumption implies that σ is independent of θ). Now consider the first auction. For any reserve price, $r\in(0,\delta p)$ where δ is the seller's discount factor, the average resale price will exceed the reserve and the seller can increase profits by raising the reserve.

The assumption of some private values is important in the argument. To see this, modify the above example by making the environment a pure common value one instead, so that $u(x,\theta)=\theta$. In this case, conditional on a failure to sell at reserve price r in the first auction, equations (1) and (2) imply that the conditional expected value of θ is below r. Since the conditional expected value of θ is an upper bound on the expected revenues in such auctions as long as bidders in the second auction are aware that the property had failed to sell in the earlier auction, we should never expected average resale prices to exceed *r*.[21]

There are several limitations of the model that should be acknowledged. We assume symmetry among the buyers. While this may be realistic for a given type of buyers, house auctions attract both buyers who desire a house to inhabit, and dealers or brokers, who will sell any properties they buy. These two types of buyers may have distinct value distributions, that is, both u and F may vary across the two classes. In addition, in our model, information collection is a discrete decision. In practice, information collection might be better modeled as a continuous variable. Moreover, we have assumed symmetry in the information collection, or participation, cost s. While we consider that constant participation cost is a better model in many applications than an exogenous set of bidders, a more general model than either case would posit a distribution of participation costs. We expect the analysis to be robust to such increasing costs, but the complexity of such a model is daunting.[22] Finally, we remind the reader that condition (4) is a sufficient condition for the result. If either participation falls with an increase in the reserve price or the screening level falls with an increase in the reserve, then the impact of a rise in the reserve may (but not must) be reversed. We believe that (4) is the most likely result.

3. An Illustrative Example

As an example of how Corollary 6 can be implemented to determine if a reserve price was too low, we collected auction information from a data set of first time sales from four auctions with published reserve prices. The data come from four oral auctions held by First

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[21] We owe a debt to an anonymous referee for inducing us to discover this implication.

[22] One reason to expect that Corollary 5 would continue to hold in a model with a distribution of participation costs is that the seller now has some monopoly power, and thus has an incentive to raise the reserve price above the socially optimal level. Thus, our analysis of the socially optimal reserve should remain a lower bound. The analysis, however, is even more complicated than the current study, for there must now be a critical level of the participation cost, so that agents with lower participation cost choose to participate.

Interstate Bank between April 1990 and September 1991 for properties throughout Texas. Although each auction was for multiple properties throughout Texas (1036 properties), our sample is from sales in Travis, Harris and Dallas counties since we obtained access to their central appraisal office records. In all four auctions, registered bidders are required to provide a $3000 deposit for each property they plan to bid on. All sales below a predetermined threshold (two auctions at $15,000 and two at $25,000) had to be purchased with all cash within 10 days. For sales exceeding such thresholds, the seller is required to provide a 5% deposit and has 30 days to close. The bidder's inability to provide with the remaining cash or financing within the time period resulted in the forfeit of his deposit. Many of these properties were poorly described in the auction brochure; there is no reason to think that a poor description in the auction listing is correlated with any other variable, but we can not rule out such a correlation (and consequent sample selection bias). In order to keep the type of objects as homogeneous as possible, we restricted attention to sales of buildings, ruling out sales of land alone, yielding a total of 26 properties offered at auction. Within this subset, the only class of properties that failed to sell were residential properties.

Of the 26 properties for which we have data, 21 sold in the auction and 5 sold later. Only residential properties failed to sell. One problem with the specific data is that, because of the way the set was constructed, while we have all the properties that did not sell at auction and which later did sell, we cannot be absolutely sure that we have listed all the properties that did not sell. If there were properties that did not enter the data set because they were never sold, our estimates of the value of unsold properties will be biased upward.

The reserve price averaged 48% of appraised value for the properties that sold, and 60% for the properties that didn't sell, suggesting that high reserve prices significantly increased the likelihood that the property failed to sell. The present discounted average sale price of properties that initially failed to sell was 93% of their appraised value [23] The average number of days to resale is 125 days. Table One gives the data for properties that failed to sell in the first auction.

Table One: First Interstate Resale Data.

Reserve	PVSP	Appraised Value	Days to Resale
140,000	178,832	182,130	365
15,000	41,340	30,879	107
20,000	30,746	45,054	60
65,000	82,422	103,630	51
20,000	25,844	30,000	44

We computed the variable *PVSP* for unsold properties using an annual interest rate of *5%*. Corollary 6 then offers a guide to test whether the reserve price that was used on properties that failed to sell was too high. The sample average of *r-PVSP* over the five unsold properties is -*$-19837* with standard error, *$6554.* The theory suggests a one-sided test of the hypothesis that *r-PVSP* is positive. Observe that in all cases, it is negative. The critical $t_{.05,n-1}$ value is *-2.13* while the sample yields a test statistic of *-3.03* suggesting, in this case, that raising the reserve price would have increased expected revenues. [24]

4. Conclusion

In a bidding model with endogenous entry, this paper demonstrates the quite intuitive conclusion that the seller should post a reserve price at least as large as, and generally strictly larger than, the average value (to the seller) of goods that fail to meet the reserve. The intuition for this conclusion rests on two observations. First, if entry into the auction is endogenous, *ex*

[23] For the properties sold by the First Interstate, we obtained the assessed value prevailing prior to the auction from county records; these are generally updated every two years.

[24] We also tested the hypothesis that *(R-PVSP)/AV* is positive. The corresponding sample mean, standard error and test statistic are *-.33*, *0.15* and *-2.3*, also generating a rejection.

ante bidder profits are zero, and thus the seller captures all the gains from trade. For this reason, the seller wishes to post a reserve that maximizes the expected gains from trade. Second, this reserve is at least the seller's alternate use value. This second observation is deceptive, for a change in the reserve price will generally alter the bidders' participation decisions, which affects the sellers' surplus. Indeed, the seller generally wishes to post a reserve strictly higher than the seller's value of items retained at the margin, because this reduces the duplication of investment in information by bidders. Under private values, the seller wishes to post a reserve between the seller's value for marginal items (where the highest bidder is just indifferent between paying the reserve and not) and the average value to the seller of items that sell at the reserve price.

In addition, we demonstrated that the lower bound is at least as large as the average value of items that fail to sell. This result seems intuitive, in that the value of items that just fail to sell at the posted reserve would presumably exceed the value of items that didn't come close to selling. However, this intuition is complicated by the fact that there are two reasons an item might fail to sell. First, a bidder considered bidding and decided the reserve was too high. The value of these items is less than the value of items at the margin of not selling. Second, an item will not sell if no bidder considered purchasing it. These items have a value distributed like the *ex ante* value, which is potentially larger than the value of items at the margin of not selling. However, under the sufficient condition for the equilibrium bidding function to be monotonic, the first reason dominates the second, and on average, items that fail to sell are worth less than those right at the margin of not selling.

We, thus, have a testable prediction: if the reserve price is less than the average value to the seller of items that fail to meet the reserve in previous auctions, raising the reserve price to the average value of unsold items will increase seller revenue on average. This prediction is also a prescription for raising seller revenue. We illustrated the theory using data on auctions with published reserve prices. The test is not as powerful as one might desire, because of limited sample size, some possibility of selection bias in data acquisition, and because of alternative explanations for expected sale prices increasing in the reserve. Nevertheless, the example suggests that increasing the reserve price will significantly increase the expected present value of sale.

We consider that the auction model with endogenous entry is a significant improvement in realism over models with exogenous participation. Endogenous entry implies that the seller maximizes revenue by maximizing *ex ante* social surplus, which simplifies parts of the analysis. However, endogenous entry also complicates the analysis, and plausible economic propositions, such as an increase in the reserve price decreasing bidder participation, appear difficult to prove in general. It seems evident that log supermodularity, so useful in environments with exogenous participation, is inadequate for environments with endogenous participation, and further work on the theory of auctions with endogenous participation is warranted. Finally, while endogenous entry represents an increase in realism, our model is hardly an exact representation of real auctions, as described by Ashenfelter (1989).

The model contains two endogenous variables, the probability of participation ρ and the screening value x_r, but we considered alterations of only one exogenous variable, the reserve price r. It is thus likely that using a second control variable, such as an entry fee, will permit better seller optimization. As a practical matter, most auctioneers do not charge entry fees, although there are notable exceptions. If optimal entry fees turn out to be negative, charging the negative entry fee is subject to a severe moral hazard problem, with people participating only in order to collect the negative entry fee. Moreover, establishing the effects of entry fees would require quantitative, rather than qualitative, information on the signs of the comparative statics in

(4). Nevertheless, an attempt to establish bounds on optimal entry fees appears to be an important unsolved problem.

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Appendix

Proof of Lemma 1: Substituting $u(x,\theta) = x$ into (2), we have $x_r = r$. From (4), $B(x)=x$. Thus, by (3):

$$\begin{aligned} s &= E_\theta\Big[\int_{x_r}^{\infty} f(x|\theta)\Big[(x-r)(1-\rho(1-F(x_r|\theta)))^{n-1} \\ &\quad + \int_{x_r}^{x}(x-y)(n-1)(1-\rho(1-F(y|\theta)))^{n-1}\rho f(y|\theta)\,dy\Big]dx\Big] \\ &= E_\theta\Big[\int_{r}^{\infty} f(x|\theta)\Big[(x-r)(1-\rho(1-F(r|\theta)))^{n-1} + (x-y)(1-\rho(1-F(y|\theta)))^{n-1}\Big|_r^x \\ &\quad + \int_{r}^{x}(1-\rho(1-F(y|\theta)))^{n-1}\,dy\Big]dx\Big] \\ &= E_\theta\Big[\int_{r}^{\infty}(1-F(x|\theta))(1-\rho(1-F(x|\theta)))^{n-1}\,dx\Big]. \end{aligned}$$

The right hand side is obviously decreasing in both r and ρ, which gives a unique solution with $x_r = r$ increasing in r, and ρ decreasing in r. ■

Proof of Lemma 2: Note that (1) implies $f(x_r|\theta)$ is nondecreasing in θ. Because of the complexity of some of the terms below, we adopt the convention that u, F and f are evaluated at (x_r,θ) and $(x_r|\theta)$ unless otherwise indicated. We also denote E_θ by E. Recall that

$$\tilde{E}(\bullet) = \frac{E(\bullet)(1-\rho(1-F))^{n-1}f}{E(1-\rho(1-F))^{n-1}f}$$

We will use the following lemma several times.

Lemma A: $\dfrac{E(1-F)^2}{(E(1-F))^2} \geq \dfrac{E(1-F)f}{E(1-F)\,Ef} \geq \dfrac{Ef^2}{(Ef)^2}$.

Proof of Lemma A:

$$E(1-F)^2 = E\tfrac{1-F}{f}(1-F)f = Ef\frac{E\frac{1-F}{f}(1-F)f}{Ef} \geq Ef\frac{E\frac{1-F}{f}f}{Ef}\frac{E(1-F)f}{Ef} = \frac{E(1-F)}{Ef}E(1-F)f.$$

$$E(1-F)f = E\tfrac{1-F}{f}ff = Ef\frac{E\frac{1-F}{f}ff}{Ef} \geq Ef\frac{E\frac{1-F}{f}f}{Ef}\frac{Ef^2}{Ef} = \frac{E(1-F)}{Ef}Ef^2. \qquad ■$$

By (2), $r = \tilde{E}u$.

$$\frac{\partial\tilde{E}u}{\partial x_r} = \tilde{E}u_x + \rho(n-1)\Big[\tilde{E}u\tfrac{f}{1-\rho(1-F)} - \tilde{E}u\tilde{E}\tfrac{f}{1-\rho(1-F)}\Big] + \tilde{E}u\tfrac{f_x}{f} - \tilde{E}u\tilde{E}\tfrac{f_x}{f} > 0.$$

$$\left.\frac{\partial \tilde{E}u}{\partial \rho}\right|_{\rho=0} = -(n-1)[\tilde{E}u(1-F) - \tilde{E}u\tilde{E}1-F] < 0.$$

Thus, $1 = \frac{\partial \tilde{E}u}{\partial x_r}\frac{dx_r}{dr} + \frac{\partial \tilde{E}u}{\partial \rho}\frac{d\rho}{dr}$, and either $\frac{dx_r}{dr} > 0$ or $\frac{d\rho}{dr} < 0$.

Eliminate r from (3) by substituting $r = \tilde{E}u$ to obtain

$$\text{(A1)}\quad S = E\int_{x_r}^{\infty} f(x|\theta)\Bigg((u(x,\theta)-\tilde{E}u)(1-\rho(1-F))^{n-1} + \int_{x_r}^{x}(u(x,\theta) - B(y))(n-1)(1-\rho(1-F(y|\theta)))^{n-2}\rho f(y|\theta)\,dy\Bigg)\,dx.$$

To establish (4), it suffices to show that $\left.\frac{dx_r}{d\rho}\right|_{S=s} < 0.$

$$\left.\frac{\partial S}{\partial x_r}\right|_{\rho=0} = -E(1-F)\left.\frac{\partial \tilde{E}u}{\partial x_r}\right|_{\rho=0} < 0.$$

Since $\left.\frac{\partial S}{\partial x_r}\right|_{\rho=0} < 0$, it suffices to show that $\left.\frac{\partial S}{\partial \rho}\right|_{\rho=0} < 0$.

$$\begin{aligned}
\left.\frac{\partial S}{\partial \rho}\right|_{\rho=0} &= E\int_{x_r}^{\infty} f(x|\theta)\Bigg[-(u(x,\theta) - \tilde{E}u)(n-1)(1-F) - \left.\frac{\partial \tilde{E}u}{\partial \rho}\right|_{\rho=0} \\
&\quad + \int_{x_r}^{x}[u(x,\theta) - B(y)](n-1)f(y|\theta)\,dy\Bigg]dx \\
&= (n-1)\tilde{E}uE(1-F)^2 - E(1-F)\left.\frac{\partial \tilde{E}u}{\partial \rho}\right|_{\rho=0} - (n-1)E\int_{x_r}^{\infty} f(x|\theta)u(x,\theta)(1-F)\,dx \\
&\quad + E\int_{x_r}^{\infty}(1-F(x|\theta))[u(x,\theta) - B(x)](n-1)f(x|\theta)\,dx \\
&= (n-1)\Bigg[[E(1-F)^2]\tilde{E}u + [E(1-F)]\big(\tilde{E}u(1-F) - \tilde{E}u\tilde{E}(1-F)\big) \\
&\quad + E\int_{x_r}^{\infty}[(F - F(x|\theta))u(x,\theta) - (1-F(x|\theta))B(x)]f(x|\theta)\,dx\Bigg] \\
&= (n-1)\Bigg[\frac{E(1-F)^2\,Euf}{Ef} + \frac{E(1-F)\,Eu(1-F)f}{Ef} - \frac{E(1-F)\,EufE(1-F)f}{(Ef)^2} \\
&\quad + E\int_{x_r}^{\infty}[(F - F(x|\theta))u(x,\theta) - (1-F(x|\theta))B(x)]f(x|\theta)\,dx\Bigg] \equiv (n-1)\varphi(x_r).
\end{aligned}$$

Note that $\varphi(\infty)=0$. Thus, to show that $\frac{\partial S}{\partial \rho}\Big|_{\rho=0} < 0$, it suffices to show that $\varphi'(x_r) \geq 0$ for all x_r.

$$\varphi'(x_r) = \frac{\partial}{\partial x_r}\left[\frac{E(1-F)^2\, Euf}{Ef} + \frac{E(1-F)\, Eu(1-F)f}{Ef} - \frac{E(1-F)\, Euf\, E(1-F)f}{(Ef)^2}\right]$$

$$+\ Ef(1-F)B(x_r) + E\int_{x_r}^{\infty} f(x|\theta)\, f u(x,\theta)\, dx$$

$$\geq \frac{\partial}{\partial x_r}\left[\frac{E(1-F)^2\, Euf}{Ef} + \frac{E(1-F)\, Eu(1-F)f}{Ef} - \frac{E(1-F)\, Euf\, E(1-F)f}{(Ef)^2}\right]$$

$$+\ Ef(1-F)\frac{Euf^2}{Ef^2} + E(1-F)fu$$

$$= -2\frac{E(1-F)f}{Ef}Euf - \frac{E(1-F)^2}{(Ef)^2}Ef_x\, Euf + \frac{E(1-F)^2}{Ef}\left[Eu_x f + Euf_x\right]$$

$$-\ \frac{E(1-F)\, Eu(1-F)f}{(Ef)^2}Ef_x + \frac{E(1-F)}{Ef}\left[Eu_x(1-F)f - Euf^2 + Eu(1-F)f_x\right] - E(1-F)fu$$

$$+\ \frac{Euf\, E(1-F)f}{Ef} + 2\frac{E(1-F)Euf\,E(1-F)f\,Ef_x}{(Ef)^3} + \frac{Ef(1-F)\, Euf^2}{Ef^2} + E(1-F)fu$$

$$-\ \frac{E(1-F)}{(Ef)^2}\left[Eu_x f\, E(1-F)f + Ef_x\, u\, E(1-F)f - Euf\, Ef^2 + Euf\, E(1-F)f_x\right]$$

$$= -\frac{E(1-F)f}{Ef}Euf - \frac{E(1-F)^2}{(Ef)^2}Ef_x\, Euf + \frac{E(1-F)^2}{Ef}\left[Eu_x f + Euf_x\right]$$

$$-\ \frac{E(1-F)\, Eu(1-F)f}{(Ef)^2}Ef_x + \frac{E(1-F)}{Ef}\left[Eu_x(1-F)f - Euf^2 + Eu(1-F)f_x\right]$$

$$+\ 2\frac{E(1-F)Euf\,E(1-F)f\,Ef_x}{(Ef)^3} + \frac{Ef(1-F)\, Euf^2}{Ef^2}$$

$$-\ \frac{E(1-F)}{(Ef)^2}\left[Eu_x f\, E(1-F)f + Ef_x\, u\, E(1-F)f - Euf\, Ef^2 + Euf\, E(1-F)f_x\right]$$

$$= \frac{E(1-F)^2\, Eu_x f}{Ef} - \frac{E(1-F)E(1-F)f\,Eu_x f}{(Ef)^2} + \frac{E(1-F)\, Eu_x(1-F)f}{Ef}$$

$$+\ \frac{Euf^2\, E(1-F)f}{Ef^2} - \frac{Euf\, E(1-F)f}{Ef} - \frac{Euf^2\, E(1-F)}{Ef} + \frac{Euf\, E(1-F)\, Ef^2}{(Ef)^2}$$

$$+\ \frac{E(1-F)^2\, Euf_x}{Ef} - \frac{E(1-F)^2\, Euf\, Ef_x}{(Ef)^2} - \frac{E(1-F)\, E(1-F)f\, Euf_x}{(Ef)^2} + \frac{E(1-F)E(1-F)f\, Euf\, Ef_x}{(Ef)^3}$$

$$+\ E(1-F)\left[\frac{E(1-F)uf_x}{Ef} - \frac{E(1-F)f_x\, Euf}{(Ef)^2} - \frac{E(1-F)uf\, Ef_x}{(Ef)^2} + \frac{E(1-F)f\, Euf\, Ef_x}{(Ef)^3}\right]$$

$$= \left[\frac{E(1-F)^2}{Ef} - \frac{E(1-F)\,E(1-F)f}{(Ef)^2}\right] E u_x f + \frac{E(1-F)\,E u_x (1-F) f}{Ef}$$

$$+ \left[\frac{Euf^2}{Ef^2} - \frac{Euf}{Ef}\right]\left[E(1-F)f - \frac{E(1-F)\,Ef^2}{Ef}\right]$$

$$+ \left[E(1-F)^2 - \frac{E(1-F)E(1-F)f}{Ef}\right]\left[\tilde{E}\,u\frac{f_x}{f} - \tilde{E}u\,\tilde{E}\frac{f_x}{f}\right]$$

$$+ E(1-F)\,\tilde{E}\left[(1-F)(u-\tilde{E}u)\left(\frac{f_x}{f} - \tilde{E}\frac{f_x}{f}\right)\right] \geq 0.$$

The first line is positive by lemma A, the second since

$$\frac{Euf^2}{Ef} \geq \frac{Euf}{Ef}\,\frac{Ef^2}{Ef}$$

and the second part of lemma A. The third is positive by lemma A and the fact that u and $\frac{f_x}{f}$ are increasing by affiliation. That the fourth is positive requires an argument. The fourth term comes in the form $E\alpha\beta\gamma$, where α, β, γ are all increasing, $\alpha>0$, and $E\beta = E\gamma = 0$. Define an expectation

$E'(\;) = \frac{E\alpha(\;)}{E\alpha}$. Then $E\alpha\beta\gamma = E\alpha\,E'\beta\gamma \geq E\alpha\,E'\beta\,E'\gamma = \frac{E\alpha\beta\,E\alpha\gamma}{E\alpha} \geq \frac{E\alpha\,E\beta\,E\alpha\,E\gamma}{E\alpha} = 0.$

This shows the fourth term is positive, as desired. Thus $\varphi'(x_r) > 0$, and

$$\frac{\partial S}{\partial \rho}\Big|_{\rho=0} = (n-1)\varphi(x_r) = -(n-1)\int_{x_r}^{\infty}\varphi'(x)\,dx < 0. \qquad \blacksquare$$

Sufficient Condition for (4):

Assuming stability implies that $\partial S/\partial\rho < 0$, it is sufficient to prove that $\partial S/\partial x_r < 0$. As before, we suppress the arguments $(x_r|\theta)$ and (x_r,θ). Using (2) and (A1),

$$\frac{\partial S}{\partial x_r} = E_\theta[(1-\rho(1-F))^{n-1}f]\left[(n-1)\rho(B(x_r)-\tilde{E}u)\tilde{E}\left(\frac{1-F}{1-\rho(1-F)}\right) - \tilde{E}\left(\frac{1-F}{f}\right)\left(\frac{\partial\tilde{E}u}{\partial x_r}\right)\right]$$

Thus, $\frac{\partial S}{\partial x_r} < 0$ if and only if

$$0 < \left[\tilde{E}\frac{1-F}{f}\right]\left[\tilde{E}u_x + \tilde{E}\left(u\frac{f_x}{f}\right) - \tilde{E}u\,\tilde{E}\frac{f_x}{f}\right] + (n-1)\rho(B(x_r)-\tilde{E}u)\left[\tilde{E}\frac{1-F}{f}\,\tilde{E}\frac{f}{1-\rho(1-F)} - \tilde{E}\frac{1-F}{1-\rho(1-F)}\right]$$

Since u_x is nonnegative, we may drop it. The resulting sufficient condition becomes

$$\text{(A2)} \quad 0 < \left[\tilde{E}\frac{1-F}{f}\right]\left[\tilde{E}\left(u\frac{f_x}{f}\right) - \tilde{E}u\,\tilde{E}\frac{f_x}{f}\right] + (n-1)\rho(B(x_r)-\tilde{E}u)\left[\tilde{E}\frac{1-F}{f}\,\tilde{E}\frac{f}{1-\rho(1-F)} - \tilde{E}\frac{1-F}{1-\rho(1-F)}\right]$$

Since (A2) is linear in u, (A2) holds if and only if (A2) holds for a basis of u. A convenient basis is the indicator functions, $u = 1$ if $\theta \geq \theta^*$, and 0 otherwise. Thus, a sufficient condition for (4) (when combined with stability) is that, for all θ^*,

$$\tilde{E}\left[\frac{f_x}{f} | \theta \geq \theta^*\right] - \tilde{E}\left[\frac{f_x}{f}\right] + (n-1)\rho\left[1 - \frac{\tilde{E}\frac{1-F}{1-\rho(1-F)}}{\tilde{E}\frac{1-F}{f}\,\tilde{E}\frac{f}{1-\rho(1-F)}}\right]\left[\tilde{E}\left(\frac{f}{1-\rho(1-F)} | \theta \geq \theta^*\right) - \tilde{E}\frac{f}{1-\rho(1-F)}\right] \geq 0.$$

This condition is also necessary for (4) to hold for all nondecreasing u.

Proof of Lemma 3: Equation (6) is a routine computation from (5) using (2). We show (7) for the case of second price auctions. The argument foran oral auction is similar but more tedious. McAfee and Vincent (1992) show the result for first price auctions. First integrate (3) by parts to obtain:

$$\text{(A3)}\; s = E_\theta\Big[(1-F(x_r|\theta))(u(x_r,\theta)-r)(1-\rho(1-F(x_r|\theta)))^{n-1}$$

$$+\int_{x_r}^{\infty}(1-F(x|\theta))\,u_x(x,\theta)(1-\rho(1-F(x|\theta)))^{n-1}\,dx$$

$$+\int_{x_r}^{\infty}(1-F(x|\theta))(u(x,\theta)-B(x))(n-1)(1-\rho(1-F(x|\theta)))^{n-2}\rho f(x|\theta)\,dx\Big].$$

Define $\hat{E}_x(\bullet) = \dfrac{E_\theta\left[(\bullet)(1-\rho(1-F(x|\theta)))^{n-2}f(x|\theta)^2\right]}{E_\theta\left[(1-\rho(1-F(x|\theta)))^{n-2}f(x|\theta)^2\right]}.$

$$\text{(A4)}\quad E_\theta\left[(1-F(x|\theta))(u(x,\theta)-B(x))(n-1)(1-\rho(1-F(x|\theta)))^{n-2}f(x|\theta)\right]$$

$$= \hat{E}_x\left[\frac{1-F(x|\theta)}{f(x|\theta)}(u(x,\theta)-B(x))\right]E_\theta\left[(1-\rho(1-F(x|\theta)))^{n-2}f(x|\theta)^2\right]$$

$$\geq \hat{E}_x\left[\frac{1-F(x|\theta)}{f(x|\theta)}\right]\hat{E}_x\left[(u(x,\theta)-B(x))\right]E_\theta\left[(1-\rho(1-F(x|\theta)))^{n-2}f(x|\theta)^2\right] = 0.$$

Integrating (5) by parts,

$$\Psi = E_\theta\Big[\sigma(\theta) - (u(x,\theta) - \sigma(\theta))\big(1 - (1-\rho(1-F(x|\theta)))^n\big)\Big|_{x_r}^{\infty}$$

$$+ \int_{x_r}^{\infty} u_x(x,\theta)\big(1 - (1-\rho(1-F(x|\theta)))^n\big)\,dx\Big] - n\rho s$$

$$= E_\theta\Big[\sigma(\theta) + (u(x_r,\theta) - \sigma(\theta))\big(1 - (1-\rho(1-F(x_r|\theta)))^n\big)$$

$$+ \int_{x_r}^{\infty} u_x(x,\theta)\big(1 - (1-\rho(1-F(x|\theta)))^n\big)\,dx\Big] - n\rho s.$$

$$\frac{\partial\Psi}{\partial\rho} = nE_\theta\Big[(u(x_r,\theta) - \sigma(\theta))(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}$$

$$+ \int_{x_r}^{\infty} u_x(x,\theta)(1-\rho(1-F(x|\theta)))^{n-1}(1-F(x|\theta))\,dx - s\Big]$$

$$\overset{(A3)}{=} nE_\theta\Big[(r-\sigma(\theta))(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}$$

$$- \int_{x_r}^{\infty}(1-F(x|\theta))(u(x,\theta) - B(x))(n-1)(1-\rho(1-F(x|\theta)))^{n-2}\rho f(x|\theta)\,dx\Big]$$

$$\overset{(A4)}{\leq} nE_\theta[(r-\sigma(\theta))(1-F(x_r|\theta))(1-\rho(1-F(x_r|\theta)))^{n-1}].$$

■

Proof of Theorem 4: First note that

$$\frac{E_\theta[\sigma(\theta)(1-\rho(1-F(x_r|\theta)))^{n-1}(1-F(x_r|\theta))]}{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}(1-F(x_r|\theta))]}$$

$$= \frac{E_\theta\left[\sigma(\theta)\dfrac{1-F(x_r|\theta)}{f(x_r|\theta)}(1-\rho(1-F(x_r|\theta)))^{n-1}f(x_r|\theta)\right]}{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}f(x_r|\theta)]} \times \frac{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}f(x_r|\theta)]}{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}(1-F(x_r|\theta))]}$$

$$= \tilde{E}\left[\sigma(\theta)\frac{1-F(x_r|\theta)}{f(x_r|\theta)}\right] \times \frac{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}f(x_r|\theta)]}{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}(1-F(x_r|\theta))]}$$

$$\geq \tilde{E}[\sigma(\theta)]\,\tilde{E}\left[\frac{1-F(x_r|\theta)}{f(x_r|\theta)}\right] \times \frac{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}f(x_r|\theta)]}{E_\theta[(1-\rho(1-F(x_r|\theta)))^{n-1}(1-F(x_r|\theta))]} = \tilde{E}[\sigma(\theta)].$$

Thus, using (4) and Lemma 3,

$$
\begin{aligned}
\frac{d\Psi}{dr} &= \frac{\partial \Psi}{\partial x_r}\frac{dx_r}{dr} + \frac{\partial \Psi}{\partial \rho}\frac{d\rho}{dr} \\
&\geq \frac{dx_r}{dr} E_\theta\,[(\sigma(\theta) - r)\,n(1-\rho(1-F(x_r|\theta)))^{n-1}\,\rho f(x_r|\theta)] \\
&\quad + \frac{d\rho}{dr} E_\theta\,[(r-\sigma(\theta))\,n(1-\rho(1-F(x_r|\theta)))^{n-1}\,(1-F(x_r|\theta))] \\
&= \frac{dx_r}{dr} \tilde{E}\,[(\sigma(\theta) - r)]\, E_\theta\,[n(1-\rho(1-F(x_r|\theta)))^{n-1}\,\rho f(x_r|\theta)] \\
&\quad + \left(-\frac{d\rho}{dr}\right) E_\theta\,[(\sigma(\theta) - r)\,n(1-\rho(1-F(x_r|\theta)))^{n-1}\,(1-F(x_r|\theta))] \\
&\geq \frac{dx_r}{dr} \tilde{E}\,[(\sigma(\theta) - r)]\, E_\theta\,[n(1-\rho(1-F(x_r|\theta)))^{n-1}\,\rho f(x_r|\theta)] \\
&\quad + \left(-\frac{d\rho}{dr}\right) \tilde{E}\,[(\sigma(\theta) - r)]\, E_\theta\,[n(1-\rho(1-F(x_r|\theta)))^{n-1}\,(1-F(x_r|\theta))].
\end{aligned}
$$

Thus $\tilde{E}\,[\sigma(\theta) - r] > 0 \Rightarrow \frac{d\Psi}{dr} \geq 0$. Since $\tilde{E}\,[\sigma(\theta)]$ is increasing in x_r and decreasing in ρ, $\tilde{E}\,[\sigma(\theta)]$ is increasing in r. Thus, increasing r from r_0 to σ_0 leaves $d\Psi/dr > 0$. ■

Proof of Lemma 5:

$$
\begin{aligned}
&\frac{E_\theta\,[\sigma(\theta)(1-\rho(1-F(x_r|\theta)))^{n-1} f(x_r|\theta)]}{E_\theta\,[(1-\rho(1-F(x_r|\theta)))^{n-1} f(x_r|\theta)]} \\
&= \frac{E_\theta\left[\sigma(\theta)\dfrac{f(x_r|\theta)}{1-\rho(1-F(x_r|\theta))}(1-\rho(1-F(x_r|\theta)))^n\right]}{E_\theta\,[(1-\rho(1-F(x_r|\theta)))^n]} \times \frac{E_\theta\,[(1-\rho(1-F(x_r|\theta)))^n]}{E_\theta\,[(1-\rho(1-F(x_r|\theta)))^{n-1} f(x_r|\theta)]} \\
&\geq \frac{E_\theta\,[\sigma(\theta)(1-\rho(1-F(x|\theta)))^n]}{E_\theta\,[(1-\rho(1-F(x|\theta)))^n]}\;\frac{E_\theta\left[\dfrac{f(x_r|\theta)\,(1-\rho(1-F(x_r|\theta)))^n}{1-\rho(1-F(x_r|\theta))}\right]}{E_\theta\,[(1-\rho(1-F(x_r|\theta)))^n]}\;\frac{E_\theta\,[(1-\rho(1-F(x_r|\theta)))^n]}{E_\theta\,[(1-\rho(1-F(x_r|\theta)))^{n-1} f(x_r|\theta)]} \\
&= \frac{E_\theta\,[\sigma(\theta)(1-\rho(1-F(x|\theta)))^n]}{E_\theta\,[(1-\rho(1-F(x|\theta)))^n]}.
\end{aligned}
$$

■

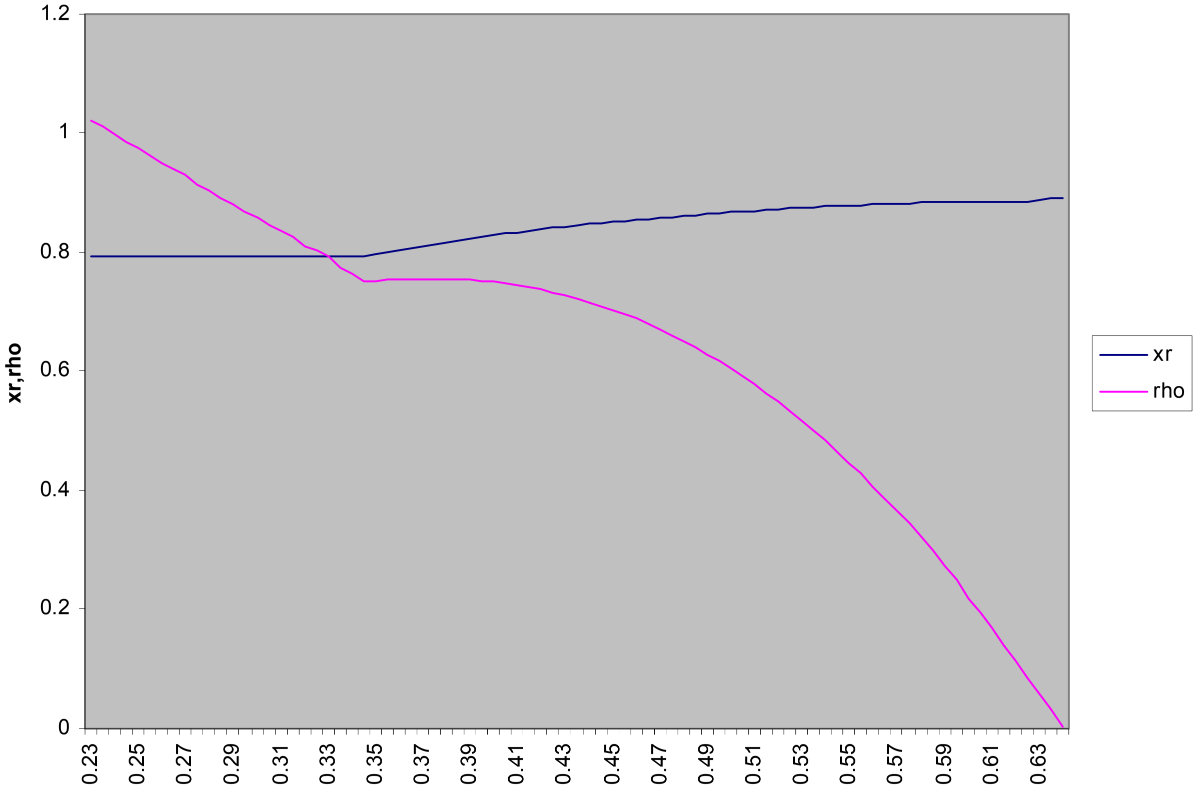

Figure 1-An example of non-monotonicity of ρ.

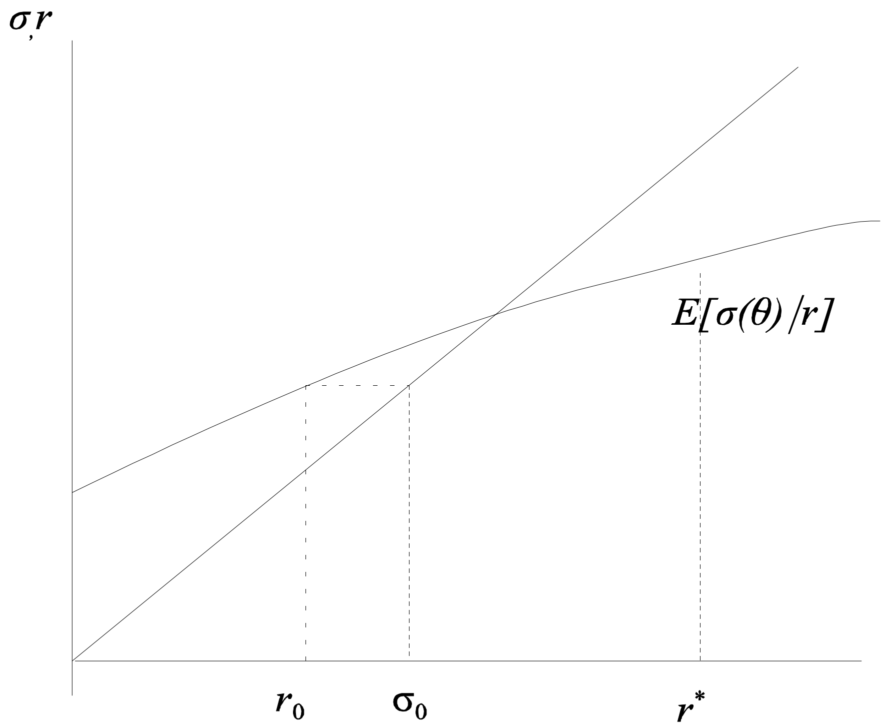

Figure 2: An illustration of Theorem 4. Given an initial reserve *r* and a realized σ_0, the reserve should be raised to at least σ_0. Note that the optimal reserve, r^*, exceeds the solution to $r = \tilde{E}[\sigma(\theta)|r]$.