Continuing Wars of Attrition

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Abstract

Asymmetries in the abilities of contestants that engage in a protracted competition appear to be more common than symmetric competitions. Why doesn't the weaker player concede immediately? This paper introduces a model based on the idea that a "war" can only be won by winning a series of battles. There are two kinds of stationary equilibria, one with fighting to completion, the other with a cessation of hostilities. As a player gets closer to losing, that player's probability of winning battles falls, social welfare rises, and the levels of effort of both players rise. The theory is applied to a variety of conflicts, including wars and attempts at market domination.

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1 Introduction

Asymmetries in the abilities of contestants that engage in a protracted competition appear to be more common than symmetric competitions. Take for instance the "browser wars" between the giant Microsoft and a fledgling Netscape in the 1990s or similarly the involvement of the US in Vietnam, the French in Vietnam and Algeria (see, e.g. Paret [1964], and Mack [1975]) or the British with the IRA during the Irish war of independence (see Kautt [1999]). In each of these there is a player (Microsoft, the US, French or the British) with a relatively large budget and a high (political) cost of a protracted war taking on weaker opponents.[1] The asymmetry is at times very acute, as exemplified by the recent travails of Percy Schmeiser against the agricultural giant Monsanto[2]. The present conflicts in Iraq and Afghanistan, as well as the earlier Afghanistan conflicts with the USSR and Britain, all have the same feature of a large foreign power fighting in an economically small region.

Who should win a war of attrition between very different players? Why doesn't the weaker player concede immediately? The examples provided above have the nature of a war of attrition, in that two sides are competing with each other for a prize that can only accrue to one of them. However, standard models fail to accommodate these examples well. First, all of the contests were decidedly asymmetric – typically with a large player and a smaller player. Second, the competitions involved one firm or country fighting for its existence against a player who survives in the event of a loss. Typically the latter player is the stronger player in resources. One might view the smaller player as budget constrained, but that doesn't seem to be a good description of Vietnam, Algeria and the US Civil War, where the budget-constrained small player somehow kept fighting. Third, the level of effort is endogenous – firms or nations can expend more or less effort at each point in time. The endogenous effort choice is important because even a small player can exert a lot of effort in the near term, perhaps inducing the larger player to exit.

This paper introduces a model capturing all three of these salient features

[1] See, for example, the statement by Gary Gallagher quoted in Zebrowski [1999] for an explanation on how the US Civil War and US/Vietnam War are analogous to the browser wars.

[2] Percy Schmeiser, an independent framer, had engaged in a protracted legal battle with agrichemical giant Monsanto between 1999 and 2004, with important implications for the farmers' rights and regulation of transgenic crops. The Canadian Supreme Court ruling which, it is generally regarded, is in favor of Monsanto can be found at http://scc.lexum.umontreal.ca/en/2004/2004scc34/2004scc34.html.

of real world competitions, based on the idea that a “war” can only be won by winning a series of battles. Victory in each battle is (endogenously) random and sways the outcome further away from the competitor’s desired position. With discounting the more distant prize becomes less desirable and increases the incentive to remain in the status quo – making it worthwhile to the weaker player to engage in the first battle. We formalize this as a discrete war of attrition.

One can think of the game as a football or soccer match along a line segment; a tug of war is the best analogy. Each player has an end or goal, between which there are a series of nodes at which a battle can occur. The object is push the battle ”front” or point of conflict to the other player’s end, just as the object in soccer is to get the ball into the other team’s goal. The first to do so wins, with the other losing. At most one step can be taken in each period; each step may be thought of as a battle. The value of winning is positive, exceeding the value of no resolution (set to zero), and the value of losing is assumed negative.

The main results are (i) that effort tends to rise as either player gets close to winning, (ii) the probability that a player advances rises the closer to winning the player is, (iii) social welfare is u-shaped, with higher utility near the goals. At any point, at least one of the players has negative utility. Moreover, in the central area, the utility of both players may be negative. This does not mean players would unilaterally exit, however, since the negative utility exceeds the utility of an immediate loss. It is possible for the outcome to be a draw, with a weakly stable interior solution. In this case, neither player wins nor loses.

Due perhaps to the origin of the formal theory of the war of attrition in evolutionary biology (Smith [1974]) and the desire for simplicity, most analyses focus on symmetric games. In all of these analyses, the effort choice is exogenous: firms either stay in, or exit. The papers in this classical vein include Fudenberg and Tirole [1984], Fudenberg and Tirole [1986], Hillman and Riley [1989], Kovenock et al. [1996], Kapur [1995], Che and Gale [1996], Krishna and Morgan [1997] and Bulow and Klemperer [1999][3] among many others.

[3] In an important paper, Jeremy Bulow and Paul Klemperer, 1999, distinguish between the IO version and the standards game of wars of attrition. In the IO version, exit stops one’s costs from accruing, while in the standards game, firms continue to incur costs until the penultimate player exits and the game ends. These two are identical in the case of two firms. Moreover, if there is only one winner, in the IO version of the game, Bulow and Klemperer prove that all but two firms drop out immediately, even when the firms are distinguished by privately known costs or values.

This paper focuses on possibly asymmetric contests with an endogenous effort choice. In the next section, a model of the war of attrition with endogenous effort is introduced. Unlike the standard models, with endogenous effort, it is necessary to keep track of the state of the system over time, because the player exerting more effort is gaining an advantage over its rival that persists. There are only a few papers that we are aware of which consider the case of endogenous effort.

Harris and Vickers [1987] was the first to study the tug of war and thus endogenize effort. The present paper builds on the analysis of McAfee [2000], the next work to study the tug of war. With respect to Harris and Vickers in 1987, there are three significant differences. Harris and Vickers use what Dixit [1987] calls a contest for the subgame, while an all-pay auction is used here.[4] Second, discounting is permitted in the present analysis, and it turns out that discounting is very important, in that the limit of the solution, as players' common discount factors converge to unity, is degenerate. These differences in modeling permit the third major difference in the analysis: a closed form solution for the stationary equilibrium, and consequently greater insight into the comparative statics of the analysis, is available with the present model.

In their analysis, Harris and Vickers emphasize the combination of strategic interaction with uncertainty. Their stage game features uncertainty in the outcome for any given levels of effort by the players. In contrast, the present study has a deterministic outcome at the stage game; the player supplying greater effort wins. Uncertainty is endogenous: the deterministic stage game outcome induces randomization in the actions of the players. Depending on the application, either model might be more appropriate.

In a recent paper, Konrad and Kovenock [2005] also study the tug of war using a first price all pay auction as the stage game. There are at least two significant differences. In their model identical effort still results in movement, with a coin toss. Here, the node has to be re-contested. Next, they identify a loss with the status quo. Here, the status-quo is strictly preferred to a loss but is less preferred to a win. Again, the application would determine which model is most appropriate. But more importantly, the differences do matter for equilibrium behavior. For instance, we exhibit "draw equilibria" where there is a region neither player finds it attractive to exert any effort. We expand on this comparison in Section 3.3.

Horner [2004] also analyzes a model in which effort choices are endoge-

[4] In addition to Dixit [1987], Grossman [1991] Garfinkel [1990] and Skaperdas [1992] provided contest models to analyze some issues considered here.

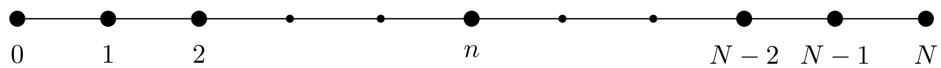


Figure 1: The playing field

nous and taken repeatedly. The structure of the model is more descriptive of a race rather than a tug of war. As such there is no immediate comparison of the results.

Section 2 presents the model and the main characterization. Section 3 examines comparitive statics, special cases and revisits applications. It also discusses possible extensions. Section 4 concludes. Most proofs are in the Appendix.

2 The Continuing War of Attrition

Two agents, named *Left* and *Right*, play a game over a set of states or nodes, indexed by $n = 0, 1, \ldots, N$. The game space is illustrated in Figure 1.

The game ends when either extreme node 0 or N is reached. Payoffs for the players are u_0 and v_0, for *Left* and *Right* respectively, when node 0 is reached, and u_N, v_N if node N is reached. Reaching the right node N is a win for *Left*, and a loss for *Right*. Conversely, reaching the node 0 is a win for *Right*. To formalize the notion of winning, assume:

$$v_0 > 0 > u_0 \quad \text{and} \quad v_N < 0 < u_N \tag{1}$$

There will be discounting and a possibility that the game never ends, resulting in a zero payoff. Thus, (1) requires that winning is preferred to delay, and delay preferred to losing. While such an assumption was not required by the standard theories, which do not involve discounting, it seems reasonable that, faced with an inevitable loss, players would prefer to delay and hence discount the loss.

At each node, the players play a first-price war of attrition by choosing non-negative effort levels. Denote by x and y the effort levels of *Left* and *Right*. The state transition is given by:

$$n \quad \rightarrow \quad \begin{cases} n & \text{if } x = y \\ n+1 & \text{if } x > y \\ n-1 & \text{if } x < y \end{cases}$$

Thus, when *Left* exerts more effort, the node is advanced, and conversely when *Right* exerts more effort. The cost of x and y are set at x and y.[5] If the game ends at time T at node $n \in \{0, N\}$, *Left*'s payoff will be of the form

$$\delta^T u_n - \sum_{t=0}^{T} \delta^t x_t$$

Right's payoff is analogous. The above game is the *continuing war of attrition* (CWA).

A strategy for a player requires a specification of an effort choice after every history, which can be arbitrarily long. However, with finitely many nodes, there will typically be a stationary equilibrium where each player's choice of effort depends only on the current state – equilibria that the literature labels Markov-Perfect. Such stationary equilibria seem natural in this context and the analysis will focus on them[6]. Furthermore if, in an equilibrium, neither player were to exert a positive effort at some node, given the transition function, play remains at that node forever. This prompts the following further classification of equilibria.

Definition 1 (Draw & No-Draw Equilibrium)**.** *An equilibrium of the CWA is said to be a draw-equilibrium if neither player exerts a positive effort at some node. Otherwise, it is said to be a no-draw equilibrium.*

Our analysis will show when these different types of equilibria occur and also offer explicit closed form solutions. The analysis begins with the stage game. Suppose *Left* and *Right* use the distributional strategies F_n and G_n at a node n. Denote by u_n and v_n the two players' continuation values at node n. Also, let $p_n(x)$ and $p'_n(x)$ respectively denote the probability of a win and a tie if *Left* bids x when *Right* bids according to G_n. Let $q_n(y)$ and $q'_n(y)$ denote similar probabilities for *Right* if she bids y. Then the following

[5] If costs are linear, setting marginal costs at unity is without loss of generality, because u_0 and u_N or v_0 and v_N can be scaled to produce an equivalent optimization problem with unit marginal cost. The equilibrium analysis holds for convex costs, provided each cost function is a scalar multiple of the other, so that rescaling produces identical costs. It is potentially important that a tie leaves the state unchanged, rather than randomly selecting another state.

[6] There may be other equilibria. In particular, there are situations where both players prefer a draw with zero effort to continued war. Moreover, a draw could be supported with a threat of a return to hostilities (positive effort) in the event that a player defects. However, this turns out to be a stationary equilibrium without resorting to history dependence.

must hold for all x, y in the support of F_n and G_n respectively:

$$\begin{aligned} u_n &= \delta p_n(x) u_{n+1} + \delta p_n'(x) u_n + \delta \left(1 - p_n(x) - p_n'(x)\right) u_{n-1} - x \\ v_n &= \delta q_n(y) v_{n-1} + \delta q_n'(y) v_n + \delta \left(1 - q_n(y) - q_n'(y)\right) v_{n+1} - y \end{aligned}$$

The above may be expressed more succinctly as

$$\begin{aligned} u_n - \delta u_{n-1} &= p_n(x) \alpha_n + p_n'(x) \alpha_n' - x \\ v_n - \delta v_{n+1} &= q_n(y) \beta_n + q_n'(y) \beta_n' - y \end{aligned}$$

where $\alpha_n = \delta(u_{n+1} - u_{n-1})$ and $\alpha_n' = \delta(u_n - u_{n-1})$ and $\beta_n = \delta(v_{n-1} - v_{n+1})$ and $\beta_n' = \delta(v_n - v_{n+1})$. α_n may be thought of as *Left*'s (net) value of winning the node n when *Right* exerts a positive effort. α_n' is the value of a tie. β_n, β_n' are similar entities for *Right*. Occasionally, we will refer to the LHS of the above equations as the net payoff of a player at that node.

The behavioral strategies at any stage in an equilibrium could occur both as mixtures or as pure strategies. When a stage equilibrium is in mixed strategies, one might expect it to correspond to the equilibrium of a standard first price war of attrition where the two players' win utilities are α_n and β_n. There is however a caveat– the payoffs from a tie are α_n' and β_n', unlike in the standard case where one player is chosen at random to be the winner if a tie occurs. As these equilibria typically involve a mass point for at least one player, taking care of the tie payoffs becomes important. The first lemma below establishes a monotonicity of payoffs which turns out to be sufficient for ensuring that behavior at each stage does seem like an equilibrium of a first price war of attrition.

Lemma 1. *For all n, $u_n \leq u_{n+1}$ and $v_n \leq v_{n-1}$, with the corresponding inequality being strict whenever $u_n \neq 0$ or $v_n \neq 0$.*

The above Lemma ensures that both players (weakly) prefer winning a node to tying, i.e. $\alpha_n \geq \alpha_n'$ and $\beta_n \geq \beta_n'$. Proposition 4, to be found in the Appendix, shows that, just as in the standard first price war of attrition, the mixed strategy equilibrium yields a zero net payoff to the lower valued player while the higher valued player gets the entire rent. Consequently, we have the following result.

Lemma 2. *The following is true of an equilibrium at any node n:*

1. *It is a pure strategy equilibrium if and only if $u_n = u_{n+1} = 0$, $v_{n-1} = v_n = 0$.*

2. It is a mixed strategy equilibrium if and only if $\alpha_n > 0$, $\beta_n > 0$ and

$$u_n - \delta u_{n-1} = \alpha_n - \min\{\alpha_n, \beta_n\} \tag{2}$$
$$v_n - \delta v_{n+1} = \beta_n - \min\{\alpha_n, \beta_n\} \tag{3}$$

From Lemma 1, we know that v_n is a decreasing sequence that is positive at $n = 0$ and negative at $n = N$. Therefore, there must be a region at the left end where $v_n > 0$ and consequently *Right* is the player with a relatively higher value, i.e. $\alpha_n < \beta_n$. Otherwise, (3) cannot hold. Accordingly, define L to be the maximal node[7] such that for all $n \leq L$, *Right* has a higher value of winning n. That is, $\alpha_n < \beta_n$ for all $n \leq L$ and $\beta_{L+1} \leq \alpha_{L+1}$. Analogous to L, one can define a minimal node, say R, such that everywhere to its right, *Left* has a (weakly[8]) higher value of winning a node, i.e. $\alpha_n \geq \beta_n$ for all $n \geq R$ and $\beta_{R-1} > \alpha_{R-1}$.

Whether *Left* values winning a node more than *Right* translates to a comparison of the sum of players' utilities at adjacent nodes. That is, letting $s_n = u_n + v_n$, $\alpha_n < \beta_n \Leftrightarrow s_{n-1} > s_{n+1}$. L and R are therefore described by the following inequalities:

$$s_{n-1} > s_{n+1} \quad \forall n \leq L \quad \text{and} \quad s_L \leq s_{L+2}, \tag{4}$$
$$s_{n-1} \leq s_{n+1} \quad \forall n \geq R \quad \text{and} \quad s_{R-2} > s_R, \tag{5}$$

Further, a direct application of Part 2 of Lemma 2 gives us the following equalities:

$$u_n = \delta u_{n-1} \qquad v_n = \delta s_{n-1} - \delta u_{n+1} \qquad n \leq L \tag{6}$$
$$v_n = \delta v_{n+1} \qquad u_n = \delta s_{n+1} - \delta v_{n-1} \qquad n \geq R \tag{7}$$

$$v_{L+1} = \delta v_{L+2} \qquad u_{L+1} = \delta s_{L+2} - \delta v_L \tag{8}$$
$$u_{R-1} = \delta u_{R-2} \qquad v_{R-1} = \delta s_{R-2} - \delta u_R \tag{9}$$

If $R \leq L + 3$, the above system of linear equations can be solved to obtain a candidate solution for v_n, u_n for all $n = 1, \ldots, N-1$. Whether this could actually constitute an equilibrium is a matter of checking whether (4)-(5) can be satisfied. On the other hand if $R > L + 3$ these equations are inadequate. Moreover the identity of the player with a relatively higher value of winning a node in $[L + 2, R - 2]$ may, in principle, switch back and

[7] Set the convention $\beta_0 > \alpha_0$ so that L is well defined.

[8] We define R with weak inequalities $\alpha_n \leq \beta_n$ merely to ensure that $L < R$ by definition.

forth several times between *Left* and *Right*. There may also be regions where neither player exerts any effort. Such multiple possibilities seem daunting at first. Fortunately, the next lemma removes much of this ambiguity.

Lemma 3. *Suppose n is such that $\beta_n \leq \alpha_n$ and $\beta_{n+1} \geq \alpha_{n+1}$.*

1. *$u_n \leq v_n$ and $u_{n+1} \geq v_{n+1}$.*
2. *If $\beta_{n-1} \leq \alpha_{n-1}$ or $\beta_{n+2} \geq \alpha_{n+2}$, correspondingly the equilibrium at n or $n+1$ is in pure strategies.*

Lemma 3 has an important implication. Recall that $\alpha_L < \beta_L$, $\alpha_{L+1} \geq \beta_{L+1}$ and $\alpha_{R-1} < \beta_{R-1}$, $\alpha_R \geq \beta_R$. If there is distinct node between $L+1$ and $R-1$, a node with a pure strategy equilibrium must occur somewhere between $L+1$ and $R-1$, which immediately leads to the following observation:

Corollary 1. *In a no-draw equilibrium, either $R = L+1$ or $R = L+3$.*

Thus, every equilibrium of the CWA divides the playing field into two zones, one on the left in which *Right* has a relatively higher value to winning a node while the opposite is true at the right end. In between, players may either settle for a draw or, in case of $R = L+3$, switch their identities for being the player with the higher valuation. The foregoing observations considerably simplify the possibilities and set the stage for a complete characterization of all equilibria.

Indeed, for $n \leq L$, u_n can be readily solved recursively using the expression given in (6) and then substituted into v_n to give

$$\begin{aligned} u_n &= \underline{u}_n \equiv \delta^n u_0 && \text{for } n \leq L, \\ v_n &= \overline{v}_n \equiv \delta^n \left(v_0 + n\left(1-\delta^2\right)u_0\right) && \text{for } n \leq L-1. \end{aligned} \quad (10)$$

Likewise,

$$\begin{aligned} v_n &= \underline{v}_n \equiv \delta^{(N-n)} v_N && \text{for } n \geq R, \\ u_n &= \overline{u}_n \equiv \delta^{(N-n)} \left(u_N + (N-n)\left(1-\delta^2\right)v_N\right) && \text{for } n \geq R+1 \end{aligned} \quad (11)$$

The functions $\underline{u}_n, \overline{u}_n, \underline{v}_n, \overline{v}_n$ are important for the analysis of equilibrium behavior and are also intuitive. $\underline{u}_n$ and $\underline{v}_n$ provide the minimum utility that the players can obtain. $\underline{u}_n$ sets out the worst that can happen to *Left*. At node n, if *Left* invests nothing in the next n battles, *Left* will lose the game n periods hence, resulting in utility $\underline{u}_n$. $\underline{v}_n$ is analogous.

Given $\underline{u}_n$, it is possible to compute *Right*'s payoff. This calculation is exactly analogous to the calculation of the higher valued player's payoff in the static first-price war of attrition. Once we know that *Left* obtains zero net utility, we can calculate his value of winning a node $\alpha_n = -\left(1-\delta^2\right)\underline{u}_n$. *Right*'s payoff can then be calculated since all the rent, $\beta_n - \alpha_n$, must accrue to her. Also note that *Right*'s payoff $\overline{v}_n$ is composed of two terms. The first term is the utility of winning, which is discounted by the minimum number of periods it will take to reach the prize. This is not to say that *Right* will reach the prize in n periods, but rather that it can, by exerting sufficient effort. The total effort exerted to win for sure, from position n, is $-\delta^n n\left(1-\delta^2\right)u_0$. In fact, the maximum effort at node m is $-\delta^m\left(1-\delta^2\right)u_0$, and discounting and summing gives the present value of the cost of effort of $-\delta^n n\left(1-\delta^2\right)u_0$. This outcome would arise if *Right* exerted maximum effort until winning.[9]

Define Ψ_n and Φ_n as

$$\begin{aligned}\Psi_n \equiv \underline{u}_n + \overline{v}_n &= \delta^n\left(v_0 + u_0 + n\left(1-\delta^2\right)u_0\right)\\ \Phi_n \equiv \overline{u}_n + \underline{v}_n &= \delta^{(N-n)}\left(u_N + v_N + (N-n)\left(1-\delta^2\right)v_N\right)\end{aligned}$$

Note that $s_n = \Psi_n$ for $n \leq L-1$ and $s_n = \Phi_n$ for $n \geq R+1$. Satisfying the constraints (4) and (5) depends on the properties of these functions. It is therefore useful to first understand their behavior to understand equilibrium behavior. It may be verified that $\delta^2\left(\Psi_{n-1} - \Psi_{n+1}\right) = \left(1-\delta^2\right)\left(\overline{v}_{n+1} - \underline{u}_{n+1}\right)$ and $\delta^2\left(\Phi_{n+1} - \Phi_{n-1}\right) = \left(1-\delta^2\right)\left(\overline{u}_{n-1} - \underline{v}_{n-1}\right)$. Moreover, $\overline{v}_n$ and $\underline{v}_n$ are decreasing while $\overline{u}_n$ and $\underline{u}_n$ are increasing. Now, if we define n_L and n_R to be the real numbers that satisfy

$$\overline{v}_{n_L} = \underline{u}_{n_L} \quad \text{and} \quad \overline{u}_{n_R} = \underline{v}_{n_R},$$

$\Psi_{n-1} > \Psi_{n+1}$ holds only if $n < n_L - 1$ while $\Phi_{n-1} < \Phi_{n+1}$ holds only if $n > n_R + 1$. Therefore, the inequalities $s_{n-1} > s_{n+1}$ and $s_{n-1} < s_{n+1}$ for $n \leq L-2$ and $n \geq R+2$ respectively that were argued to be necessary for equilibrium behavior (See (4) and (5)) can hold if and only if

$$L - 1 \leq n_L \quad \text{and} \quad R + 1 \geq n_R. \tag{12}$$

The above presents a simple necessary condition for ruling out the existence of a no-draw equilibrium, namely $n_L < n_R$. An equilibrium in this

[9] It is not an equilibrium for *Right* to do so; *Right* must randomize. If it turned out, however, that the outcomes of the randomizations were the maximum of the supports, then the outcome described arises, which gives *Right*'s utility.

case, if it exists, would necessarily be a draw-equilibrium. In fact, we have a sharper result. Let L_0 and R_0 denote integers such that

$$\Psi_{L_0-1} > 0 > \Psi_{L_0} \ \text{ and } \ \Phi_{R_0} < 0 < \Phi_{R_0+1}.$$

Note that L_0 and R_0 occur around the point where $\Psi_n = 0$ and $\Phi_n = 0$. Therefore, necessarily, $L_0 < n_L$ and $R_0 > n_R$.

Theorem 1 (Draw Equilibrium)**.** *A draw equilibrium exists if and only if* $R_0 \geq L_0 + 3$. *Moreover,*

1. *Such an equilibrium is unique with* $L = L_0$ *and* $R = R_0$.
2. *(10) and (11) give the payoffs for* $n \leq L-1$ *and* $n \geq R+1$*, while* $v_n = u_n = 0$ *for* $n = L+2, \dots R-2$. *The remaining payoffs are* $v_{L+1} = u_{R-1} = 0$ *and*

$$\begin{aligned} u_{L+1} &= -\frac{\delta^2}{1-\delta^2}\Psi_{L-1} & u_R &= \frac{\delta}{1-\delta^2}\Phi_{R+1} \\ v_L &= \frac{\delta}{1-\delta^2}\Psi_{L-1} & v_{R-1} &= -\frac{\delta^2}{1-\delta^2}\Phi_{R+1} \end{aligned} \tag{13}$$

3. *Neither player exerts any effort at* $L_0+2, \dots R_0-2$.

Part 3 of the above Theorem is especially noteworthy. A mixed strategy equilibrium is played at $L+1$. There is (of course) a positive probability of transition from a no-draw region to a draw region. But note that $\beta_{L+1} < \alpha_{L+1}$. As a result, there is a higher probability of movement to the right, i.e. to the draw region. As will be shown, the probability that *Right* wins a node increases as one moves closer to the left end, except that at $L+1$, it is reversed. The behavior of *Left* can therefore be interpreted as a "last ditch" attempt by her to force a draw instead of a likely loss.

As already mentioned, when $n_L \leq n_R$, a draw-equilibrium must exist. From Corollary 1 and (12), this is also a simple condition for non-existence of a no-draw equilibrium. A condition more precise than the mere requirement $n_R < n_L$ is needed to characterize the existence of a no-draw equilibrium.

Define n^* by $\Psi_{n^*} = \Phi_{n^*}$, with $n^* = 0$ if $\Psi_0 \leq \Phi_0$ and $n^* = N$ if $\Psi_N \geq \Phi_N$. If $n_L \leq n_R$ or $n^* \notin [n_R, n_L]$ n^* is said not to exist. Essentially, n^* exists if either Ψ_n and Φ_n do not intersect between 0 and N, or intersect in an interval where $\Psi_{n-1} - \Psi_{n+1}$ or $\Phi_{n+1} - \Phi_{n-1}$ is positive.

Theorem 2 (No-draw Equilibrium)**.** *A no-draw equilibrium does not exist if* n^* *does not exist. If* n^* *does exist however, a no-draw equilibrium with* $R = L+1$ *exists if and only if* $\Psi_{L-1} \geq \Phi_{L+1}$ *and* $\Psi_L \leq \Phi_{L+2}$. *Moreover, if* $\Psi_{n^*} \geq 0$*, this is the unique equilibrium.*

In a no-draw equilibrium where $R = L+1$, the only undetermined payoffs are v_L and u_{L+1} – all others are given by (10) and (11). The corresponding equations for these variables, from (6) and (7) are $v_L = \delta\Psi_{L-1} - \delta u_{L+1}$ and $u_{L+1} = \delta\Phi_{L+2} - \delta v_L$, which can be solved to get

$$v_L = \frac{\delta}{1-\delta^2}\left(\Psi_{L-1} - \delta\Phi_{L+1}\right) \qquad u_{L+1} = \frac{\delta}{1-\delta^2}\left(\Phi_{R+1} - \delta\Psi_{L-1}\right) \quad (14)$$

We end this section with three caveats. First, the integers L_0 and R_0 are well defined for a generic set of parameters and makes Theorem 1 a tight characterization. When $n^* \in (n_R, n_L)$ and $\Psi_n(n^*) \geq 0$, the existence of a draw equilibrium is precluded. For, in this case $L_0 > R_0$ necessarily. To guarantee the existence of a no-draw equilibrium where $R = L+1$, according to the above result, an integer L must be be found such that $\Psi_{L-1} \geq \Phi_{L+1}$ and $\Psi_L \leq \Phi_{L+2}$, which implies that $n^* \in [L, L+1]$. However, even if n^* exists, these inequalities cannot be ensured due to integer problems nor can they be resolved through a perturbation of parameters. A no-draw equilibrium with $R = L+1$ may fail to exist. Further, it turns out that a no-draw equilibrium with $R = L+3$ cannot exist when $\Psi_{n^*} \geq 0$. As a result, with $\Psi_{n^*} \geq 0$, the non-existence of an equilibrium remains a possibility. We overlook these issues that arise primarily due to the integer problems, and partly justify the rest of the analysis by looking at the continuous limit at $N \to \infty$ in Section 2.1.

Second, our two theorems do not discuss no-draw equilibria with $R = L+3$. There are two facts of such equilibria that are of economic interest. Other than this, such equilibria offer no further insights but developing an explicit set of necessary and sufficient conditions for its existence involves extremely cumbersome algebra. An interested reader may choose to consult the proof of Case 2, Theorem 1 in McAfee [2000] for the details.

The first interesting aspect of a no-draw equilibrium with $R = L+3$ is that the sum of players' payoffs at $n = L, \ldots, L+3$ is negative as are the payoffs of each player at $L+1, L+2$. This indicates a prisoner's dilemma feature to equilibria. If dropping out of the game is permitted, then the players would like to do this, but not if dropping out means losing the war. On the other hand, it is worthwhile to note that the continuation payoff in a draw equilibrium is always non-negative. The other aspect is that, the negative players' utility also implies that $\Psi_{n^*} < 0$ and as a result, with enough nodes, $L_0 < R_0$. This implies that a draw equilibrium typically co-exists with a no-draw equilibrium.

Third, we have omitted a description of the equilibrium strategies in the statements of the two Theorems. However, recall that the mixed strategy equilibrium at any node corresponds to the equilibrium of an all-pay first price auction. Therefore, at any such node, the higher valued player chooses a uniform distribution on $[0, \min\{\alpha_n, \beta_n\}]$, the lower valued player bids 0 with probability $1 - \min\{\alpha_n, \beta_n\} / \max\{\alpha_n, \beta_n\}$ and according to a uniform distribution on $[0, \min\{\alpha_n, \beta_n\}]$ with the remaining probability.

2.1 Continuum of Battles

A number of interesting conclusions can be drawn on the basis of Theorem 1 & Theorem 2 that we address in the sequel. For some of these, it is instructive to look at the limiting case when the number of battles becomes infinitely large so that in the limiting model, battles are fought along a continuum. We do this by fixing the amount of discounting required to cross the entire playing field, so that the set of points is refined while holding the overall distance constant. That is, we take

$$\mathrm{e}^{-\gamma} = \delta^N \tag{15}$$

and then send N to infinity. It is unnecessary to reduce the costs of conflict, since that is equivalent to scaling utilities. Let $\lambda = \lim_{N\to\infty} n/N$. Ψ_n and Φ_n then converge, pointwise, to

$$\begin{aligned}\Psi(\lambda) &= \mathrm{e}^{-\gamma\lambda}(u_0 + v_0 + 2\gamma u_0 \lambda)\\ \Phi(\lambda) &= \mathrm{e}^{-\gamma(1-\lambda)}(u_N + v_N + 2\gamma v_N(1-\lambda))\end{aligned}$$

Let λ_L and λ_R denote the solutions to $\Psi(\lambda) = 0$ and $\Phi(\lambda) = 0$ respectively. Then in the limit, $L_0/N \to \lambda_L$ and $R_0/N \to \lambda_R$. Further, if we let λ_L^* and λ_R^* denote the minima of Ψ and Φ, $n_L/N \to \lambda_L^*$ and $n_R/N \to \lambda_R^*$ as $N \to \infty$. From this, the findings of Theorem 1 and Theorem 2 can be summarized as follows in terms of these parameters.

1. A draw equilibrium exists if and only if $\lambda_L < \lambda_R$. Everywhere to the left of λ_L, *Right* is the player with a higher value while the opposite holds to the right of λ_R. A draw occurs in the region (λ_L, λ_R).
2. For a no-draw equilibrium, one of the following must occur:
   (a) $\Psi(\lambda^*) = \Phi(\lambda^*)$ for some $\lambda^* \in [\lambda_R^*, \lambda_L^*]$ and $0 < \lambda^* < 1$.
   (b) $\Psi(0) \leq \Phi(0)$.
   (c) $\Psi(1) \geq \Phi(1)$.

   In this case, *Right* is the player with higher value to the left of λ^*, the opposite holds to the right of λ^*.

For clarity, a situation where λ^* does not exist is depicted in Figure 2. Here the curves $\Psi(\cdot)$ and $\Phi(\cdot)$ intersect in a neighbourhood where the former is increasing. In a neighbourhood of λ^* it is impossible to satisfy $\Psi_{n-1} > \Psi_{n+1}$. Therefore a no-draw equilibrium cannot exist. In Figure 4 or Figure 5, we have a situation where an interior λ^* exists since $\Psi(\cdot)$ is decreasing and $\Phi(\cdot)$ is increasing at the point of their intersection. Therefore a no-draw equilibrium can exist.

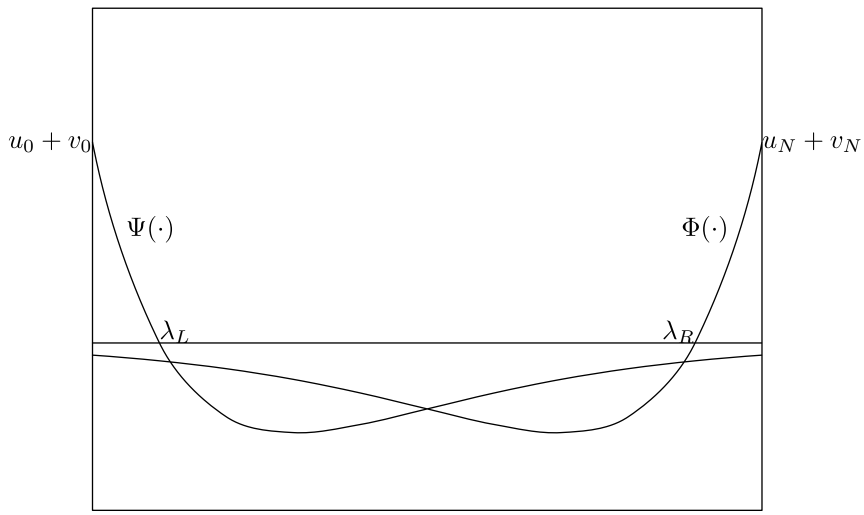


Figure 2: A case where λ^* does not exist

It may be confirmed through a routine calculation that

$$\lambda_L = -\frac{1}{2\gamma}\left(1 + \frac{v_0}{u_0}\right) \qquad 1 - \lambda_R = -\frac{1}{2\gamma}\left(1 + \frac{u_N}{v_N}\right)$$
$$\lambda_L^* = \frac{1}{2\gamma}\left(1 - \frac{v_0}{u_0}\right) \qquad 1 - \lambda_R^* = \frac{1}{2\gamma}\left(1 - \frac{u_N}{v_N}\right)$$

Figure 3 depicts the regions in which the draw and no-draw equilibria exist in the $-v_0/u_0, -u_N/v_N$ space.

Everywhere above the shaded region, $\lambda_L > \lambda_R$. Therefore in this region, a unique no-draw equilibrium occurs. Below the line with the intercept $2(\gamma - 1)$, $\lambda_L^* < \lambda_R^*$. In this region only a draw equilibrium can exist (uniquely). Between these lines, it is possible for both the no–draw equilibrium and a draw equilibrium to exist. In general, for a given set of end values, greater impatience i.e. high γ favors a draw equilibrium while a low γ, say in $(0, 1)$ rules out the uniqueness of a draw equilibrium.

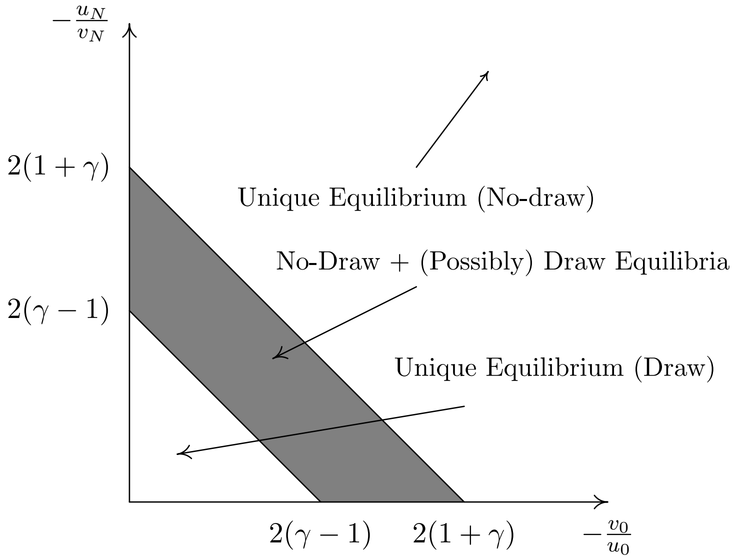


Figure 3: Regions where different equilibria exist

2.2 Effort and Welfare Levels

The sum of the utilities of the two players everywhere to the left of L is given by Ψ_n and everywhere to the right of R by Φ_n. Therefore in the limit of no-draw equilibria, the sum of players utilities is given by the upper envelope of $\Psi(\cdot)$ and $\Phi(\cdot)$ as depicted in Figure 4.

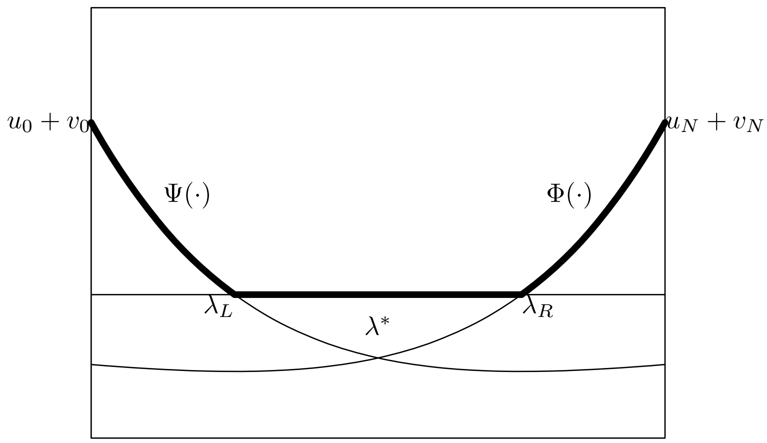


Figure 4: Sum of players' payoffs in a draw equilibrium. Draw occurs in (λ_L, λ_R).

The sum of players' utilities in the case of a draw equilibrium equals zero in the region of a draw but is given by either Ψ_n in the left field or Φ_n in the right field. In the limit of draw-equilibria, the sum of players' utilities is $\max\{\Psi(\lambda), \Phi(\lambda), 0\}$. These are depicted in Figure 4. Both in a draw and

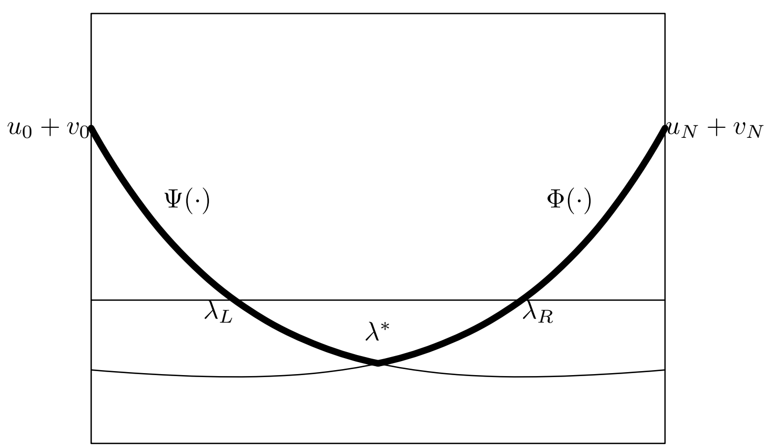


Figure 5: Sum of players' payoffs in a no-draw equilibrium

a no-draw equilibrium, the sum of players' utilities increases as we move to either end

Next, consider the maximum effort at node $n < L$. *Left*'s win value here is $\alpha_n = \delta\left(\underline{u}_{n+1} - \underline{u}_{n-1}\right) = -\left(1-\delta^2\right)\delta^n u_0$. As *Left* is the player with a lower value of winning these nodes, α_n is also the maximum effort that either player will choose at these nodes, which is seen to be increasing as one moves closer to the left end. As must be the case, the maximum effort at a node converges to zero as N gets large[10].

We collect the observations of this (sub)section in the form of a proposition below, without offering a further explicit proof.

Proposition 1. *In any equilibrium,*

1. *The sum of utilities, average and maximum efforts initially decline in n for $n < L$ and then rise for $n > R$, (possibly with a segment constant at zero).*
2. *In the limit, the total utility at λ is given by $\max\{\Psi(\lambda), \Phi(\lambda)\}$ in case of a no-draw equilibrium and by $\max\{\Psi(\lambda), \Phi(\lambda), 0\}$ in case of a draw-equilibrium.*

In words, total utility is maximized on the edges, and minimized in the center.

2.3 Probability of Win and Duration Of War

Let p_n denote the probability of transition $n \to n-1$, which is the probability of *Right*'s win. For $n < L$, *Right* is the higher valued player and from the analysis of the first price war of attrition (see Proposition 4), $p_n = 1-\alpha_n/2\beta_n > 1/2$. Making the substitution from the equilibrium payoffs for $\beta_n = \delta(\overline{v}_{n-1} - \overline{v}_{n+1})$ and $\alpha_n = \delta\left(\underline{u}_{n+1} - \underline{u}_{n-1}\right)$,

$$p_n = 1 - \frac{1}{2}\frac{-u_0}{v_0 + (n-1)(1-\delta^2)u_0 - 2\delta^2 u_0} \qquad n < L \qquad (16)$$

and, taking the limit as $n/N \to \lambda$, $N \to \infty$, $p(\lambda)$ the flow probability of *Right* winning a "battle" when there are a continuum of battles is given by

$$p(\lambda) = 1 - \frac{1}{2}\frac{-u_0}{v_0 - 2(1-\gamma\lambda)u_0}$$

where $\lambda < \lambda_L$ in a draw-equilibrium and $\lambda < \lambda^*$ in a no-draw equilibrium. For $n > R$ and $\lambda > \lambda^*$ or $\lambda > \lambda_R$, the expressions for p_n and $p(\lambda)$ can be analogously derived.

Note that p_n is only the probability of winning a node. How does this translate into the probability of winning the war? First consider a no-draw

[10]However the flow rate of effort must still be positive.

equilibrium and let q_n denote the probability that *Left* wins the war, that is, the state reaches N. The probabilities q_n are defined by $q_0 = 0$, $q_N = 1$ and

$$q_n = p_n q_{n-1} + (1 - p_n)\, q_{n+1} \tag{17}$$

Equation (17) expresses the law of motion for translating the likelihoods of winning battles into ultimate victory in the war. For economy of expression, it is configured so that p_n is the probability *Left* loses a battle, but q_n is the probability *Left* wins the war. Equation (17) states that the likelihood of winning from state n is the likelihood of winning from state $n-1$, times the probability of reaching that state, plus the likelihood of winning from state $n+1$ weighted by the probability of transition to that state.

The above expression for q_n can be rewritten to get

$$q_{n+1} - q_n = \frac{p_n}{(1 - p_n)}\,(q_n - q_{n-1}) \tag{18}$$

The above equality inducts to give

$$\begin{aligned} q_{n+1} - q_n &= \frac{p_n}{1 - p_n}\,(q_n - q_{n-1}) \\ &= (q_2 - q_1) \prod_{j=1}^{n} \frac{p_i}{1 - p_i} \end{aligned}$$

Since $q_1 = (1 - p_1)\, q_2$, the above RHS is non-negative and shows immediately that $q_{n+1} \geq q_n$. Moreover, when $n < L$, $p_n > 1/2$ and from (18), $q_{n+1} - q_n > (q_n - q_{n-1})$ while for $n > R$, the opposite inequality holds, which is to say, that q_n is convex for $n < L$. Similarly, it is concave for $n > R$.

The foregoing discussion is presented as a proposition below, again, the proof is evident.

Proposition 2. *In any equilibrium,*

1. *The probability that* Left *wins a battle is less than* 1/2 *for $n < L$ and greater than 1/2 for $n > R$.*

2. *The probability that* Left *wins a battle is increasing in n, for $n < L$ and $n > R$.*

3. *In a no-draw equilibrium, the probability that* Left *wins the war, q_n, is non-decreasing in n. In addition, q_n is convex for $n < L$ and concave for $n > R$.*

Proposition 2 has the following interesting implications. There is a defense disadvantage–the agent closest to losing the war is more likely to lose any given battle. In particular, when the battlefront is near *Left*'s home base, *Left* wins the next battle with probability less than 50%. In spite of this likelihood of losing, the closer the current node gets to the end, the harder both sides fight. Finally, there is a momentum effect. As *Left* gets closer to winning, it's likelihood of winning the next battle rises. This effect is a consequence of discounting, and doesn't arise if the players do not discount future payoffs.

Part 3 of the above result also shows that the probability *Left* wins the war is non-decreasing as one moves closer to her favorite end. While this result is only stated for the case of a no-draw equilibrium, the computations given above admit a straightforward modification to account for a draw equilibrium. In a draw equilibrium, q_n is the probability of a *draw* when $n < L$. For $n > R$, q_n is the probability of *Left*'s winning the war. It monotonicity and concavity properties are preserved in these regions. Thus, in particular, the probability that *Left* does not *lose the war* is always non-decreasing.

Yet another object of interest is the duration of the war. Let Δ_n denote the expected duration of the war. Analogous to q_n, the expected duration satisfies

$$\Delta_n = 1 + p_n \Delta_{n-1} + (1 - p_n)\, \Delta_{n+1} \tag{19}$$

This too an be analyzed analogously to q_n, but the actual solution is complicated. The following describes its behavior when N diverges.

Proposition 3. *In the limit of no-draw equilibria where $n/N \to \lambda$, $N \to \infty$,*

$$q_n \to q(\lambda) = \begin{cases} 0 & \text{if } \lambda < \lambda^* \\ 1 & \text{if } \lambda > \lambda^* \end{cases} \qquad \frac{\Delta_n}{N} \to \begin{cases} \int_0^\lambda \frac{1}{2p(x)-1} \mathrm{d}x & \text{if } \lambda < \lambda^* \\ \int_\lambda^1 \frac{1}{1-2p(x)} \mathrm{d}x & \text{if } \lambda > \lambda^* \end{cases}$$

With a very fine grid, the likelihood of winning any particular battle for *Left* converges to a number which is not zero or one, but in between. However, since the war is now composed of a very large number of battles, the outcome of the war is deterministic. The duration of the war converges to a remarkably simple expression, and is roughly linear in λ[11]. The duration can be interpreted as follows. With a likelihood of moving left of p_n, the expected net leftward movement per period is $p_n - (1 - p_n) = 2p_n - 1$. Thus,

[11] To be precise, duration is exactly linear when there is no discounting, i.e. $\gamma = 0$. In this case, p is constant (although a different constant on either side of λ^*).

$1/(2p_n - 1)$ is the expected number of periods to move one unit to the left. The integral to the current period gives the number of periods to reach zero. The analogous calculation holds to the right of λ^*

3 Comparative Statics and Special Cases

3.1 Lower Cost of Effort

In a standard war of attrition (oral first price or second price), conditional on observing a war of attrition, the lower cost player is more likely to drop out. This defect arises for the usual reason with mixed strategy equilibria – each player randomizes in such a way as to make the other player indifferent. As a consequence, the low cost player must be more likely to drop out in the next instant, so as to make it worth the cost to the high cost player of remaining in the game.[12] Fudenberg and Tirole [1986] note this defect, describing it as a consequence of mixed strategies, without additional comment.

From an economic perspective, the defect in the theory arises because the low cost player is forbidden by assumption from fully exploiting its low cost. The low cost player might like to present a show of force so large that the high cost player is forced to exit, but the usual game prohibits such endogenous effort. In most actual wars of attrition, players have the ability to increase their effort, so as to force the other side out. The US theory on war since Vietnam is that the public won't stand for a protracted conflict, and thus the US will lose if it does not win quickly. As a consequence, the US brings an overwhelming force to a conflict. (See, e.g. Correll [1993].) The 1991 Desert Storm conflict appears to be an example of this approach. Similarly, Barnes and Noble entered Internet book sales aggressively, with a large commitment of resources.

In contrast to the standard model, the continuing war of attrition does have the low cost player more likely to win, and even more likely the lower is the player's cost. This can be seen by first noting that a lower cost of effort is equivalent to rescaling both the utility of winning and losing by a factor exceeding unity. For example, if *Left*'s cost of effort is reduced in half, the effect is the same as doubling u_0 and u_N. Lowering the cost of *Left*'s effort shifts Ψ down and Φ up, and thus shifts n^* (or λ^*) to the left unambiguously.

[12] It might be that appropriately chosen refinements insure the high cost player drops out more often at the start of the game. However, having the high cost player drop out initially doesn't solve the application problem. The theory still predicts that, when a war of attrition is observed, it is the low cost player who is more likely to exit, and the high cost player more likely to win the war.

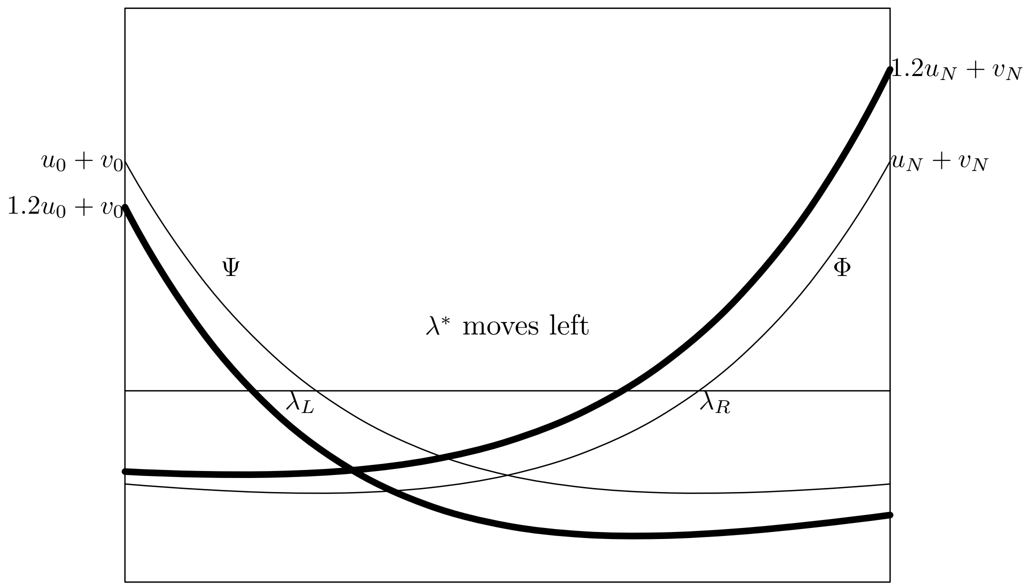


Figure 6: The effect of a 20% decrease in *Left*'s cost.

The region where *Left* is more likely to win expands. The effect on λ^* is illustrated in for the continuous case. In addition, the likelihood that *Left* wins any particular battle also increases, as we see from (16). In the limit, the duration of the war falls when *Left* is the likely winner, and rises when *Left* is the likely loser.

3.2 Patient Players

When players can take actions very often, the time between battles is reduced and the discount factor converges to 1. This is a different thought experiment than making the playing field continuous while holding the discounting required to cross the entire playing field constant. It is easy to see that $\Psi_n \to u_0 + v_0$ and $\Phi_n \to u_N + v_N$, so that there is a global winner, and it is the agent whose victory provides the greatest combined surplus. If that combined surplus is negative for either winner, then there will also be an equilibrium where there is a draw, except on the edges.

If the original game (ignoring effort) is constant sum, so that $u_0 + v_0 = u_N + v_N$, so that the combined surplus is the same for both players, then the switch point n^* satisfies[13]:

$$\frac{n^*}{N} = \frac{u_N - v_N}{u_N - v_N + v_0 - u_0}$$

Thus, the relative profits from victory determine the critical point, and *Left* is more likely to win the larger is his profits.

3.3 Zero loser utility

We have thus far focused on situations where a status quo is valued to a loss. In some cases, the disutility from a loss may not be any different from the status quo. Although we do not explicitly allow for $u_0, v_N = 0$, it is nonetheless possible to conduct the thought experiment on the limit of equilibria as $u_0, v_N \to 0$, keeping v_0 and u_N fixed. In this case, the resulting values of $-v_0/u_0$ and $-u_N/v_N$ are large putting us in the region (in terms of Figure 3) where only a unique draw equilibrium is a possibility: everywhere to the right (left) of λ^* *Left* (*Right*) is the more (less) likely winner.

Konrad and Kovenock [2005] study a tug-of-war in which the loser utility and the draw utility are zero. However, in their model, a coin toss determines the winner if the chosen efforts result in a tie. In stark contrast to the feature of zero loser utility described above, in the equilibrium of their model at most two battles are fought at a pair of adjacent nodes. At all other nodes a coin toss determines the movement. Here, on the other hand, the possibility of a draw forces every node to be contested. However, *Left*'s value of winning a node for $n < n^*$, which is $\alpha_n = -\left(1 - \delta^2\right) \delta^n u_0$ also becomes small. Therefore the probability that *Right* wins these nodes becomes close to one. (See (16)). Consequently, whoever wins the battle around the switch point, becomes the eventual winner with a probability arbitrarily close to one as[14] $u_0, v_N \to 0$.

[13] For $u_0 + v_0 = v_N + u_N$, rewrite $\Psi_n = \Phi_n$ as

$$\frac{\left(\delta^n - \delta^{N-n}\right)(u_0 + v_0)}{1 - \delta^2} = -\delta^n n u_0 + \delta^{N-n} (N - n) v_N.$$

Using L'Hopital rule and taking the limit as $\delta \to 1$ gives $-(n - (N - n))(u_0 + v_0)/2 = -\,\mathrm{nu}_0 + (N - n) v_N$, which n^* solves.

[14] The speed with which v_N converges to zero relative to the speed of convergence of u_0 makes a difference. The comments here assume they are converging at the same speed.

4 Applications

4.1 The Colonial Power

Contests such as the US in Vietnam, France in Algeria or Microsoft versus Netscape may be considered as analogous to a colonial war; one side continues to survive after a loss in the conflict, while the other is extinguished. We let the larger power be Right, with the defenders being *Left*. A victory for Right implies that *Left* is extinguished; the cost to *Left* of a loss should be viewed as being very large. This willingness to suffer any consequence to avoid losing might be modeled as $u_0 \to -\infty$. As $u_0 \to -\infty$, so does Ψ and thus the region where Right wins disappears. Here, unless the colonial power wins an instant victory, it loses.

However, the desire of the defenders not to lose is not the only salient aspect of a colonial war. The colonial power typically has a lower cost of materials, and perhaps even of troops, given a larger population. Lowering the cost of fighting is the same as a rescaling of the values of winning and losing. Thus, sending the cost of fighting to zero sends both v_0 and $-v_N$ to ∞. As we saw above, this favors the colonial power, and the prediction of the likely winner turns on whether the cost of the colonial power is relatively low, when compared with the cost of the defenders.

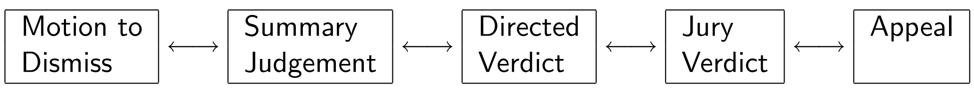
4.2 Legal Battles & Debates

Consider a tort dispute between a potentially injured plaintiff and a potentially liable defendant[15]. We let the plaintiff be *Left* and the defendant be *Right*. An important characteristic of the legal system is that a win for the defendant involves the same payoff as a draw without fighting; that is, the defendant pays the plaintiff nothing. Formally, in the model, $u_0 = v_0 = 0$, implying $\Psi_n = 0$. The prediction of the theory is that there are two regions, with a draw on the left, and fighting on the right. Thus, rather than plaintiffs formally losing, the plaintiffs just go away when the situation favors the defendant.

If $u_N + v_N < 0$, then a draw is the unique outcome. At first glance, it might seem that a legal dispute has to be a negative sum game. However, these values are scaled by the cost of effort, so that, when the plaintiff has a lower cost of effort, u_N may well exceed $-v_N$, even when the original game is zero sum.

[15] Litigation is often viewed as a war of attrition. For a particularly entertaining example, see Supreme Court [1997] for an account of David Kim's battles with Sotheby's auction house, at Supreme Court of New York.

In order to win, a plaintiff needs to survive a number of challenges by the defendant. First, defendants regularly challenge the plaintiff's standing to sue. If the plaintiff survives, the defendant requests summary judgment – that the plaintiff can't, as a matter of law, win. If the plaintiff wins that battle, the plaintiff is permitted to put on a case. At this point, the defendant typically requests a directed verdict, alleging that the plaintiff has failed to prove their case. Again, should the plaintiff prevail, the defendant puts on their case; if the plaintiff prevails, typically the defendant appeals. One can think of this as a five node battle (this means N=6).



If the plaintiff loses at a stage, the plaintiff can appeal; victory in the appeal permits advancement to the right[16]. As a practical matter, if the plaintiff loses the jury verdict, appeal is relatively difficult. In the model, the current node is a sufficient statistic for the state of the system. In an actual legal conflict, as in many conflicts, history will matter – even when an appeal sets aside a jury verdict, some of the issues (such as discovery limits) may remain.

4.3 Lobbying

Consider a collective choice environment where the policy space is a line segment with two players endowed with Euclidean preferences. Suppose the current status-quo policy is x_S and that *Left* and *Right* have a bliss point at x_L and x_R respectively, where $x_L < x_S < x_R$. Suppose a win by a player means switching the collective decision to her favorite policy. Typically a "contest" such as the one introduced in Tullock [1980] (and analyzed further in Dixit [1987]) is used to study rent seeking behavior in such environments. Instead, it is reasonable to posit that a player must win a series of battles before winning the decision to be in her favor. The implications of modelling this as a CWA are as follows.

Normalizing the utility of the status quo to zero, $v_0 = r$, $v_N = r - t$, $u_0 = \ell - t$ and $u_N = \ell$ where $\ell = (x_S - x_L)$, $r = (x_R - x_S)$ and $t = (x_R - x_L)$. That is, r for *Right* and ℓ for *Left* are the option values of switching to her favorite policy. t determines the utility loss if the competitor's favorite outcome is selected.

[16]In the event of a victory, the plaintiff can sell the rights to the judgment to the Judgment Purchase Corporation, who will then handle the appeal. See Fisk [1999]

Note that

$$-\frac{v_0}{u_0} - \frac{u_N}{v_N} = \frac{r}{t-\ell} + \frac{\ell}{t-r} \qquad 0 \leq \ell, r < t$$

Whether the two players choose to necessarily fight until a change is adopted or perhaps settle for a draw can be seen from Figure (3) on examining the above sum. When either r or ℓ is close to t, the above sum is high. This describes a situation where the status quo is close to the favorite policy of at least one player. These situations are characterized by a unique no-draw equilibrium in which the players fight incessantly until one player wins. When players are similar so that both r and ℓ are close to each other, a draw equilibrium obtains. In the intermediate range, both a draw and a no-draw equilibrium occurs.

5 Extensions

5.1 A minimum level of effort

In many situations, it is reasonable to assume that a certain minimum effort must be exerted before play from a node can be advanced in either direction. It turns out that allowing for this modification can result in a qualitative change in equilibrium behavior. Indeed, introduce $m > 0$ so that actions in the stage game lie in $0 \cup [m, \infty]$. A transition $n \to n \pm 1$ cannot be effected unless the effort choices x, y satisfy $\max\{x, y\} \geq m$. Otherwise, the game is identical to the one studied earlier.

A minimum level of effort produces an important feature of equilibrium not present in the previous analysis. There can be uncontested regions, where one player exerts a minimum level of effort and the other exerts none. These regions tend to surround a draw region, where neither player exert effort. Moreover, the player who fails to contest in the uncontested region eventually starts fighting to delay the end of the game, but in the uncontested region, the end is sufficient distant that it is not worth the minimum to delay further. Another new feature that emerges here is the impossbility of a transition to a draw region from a region where fighting occurs. The minimum can also kill no-draw equilibria.

For brevity, we provide a sketch of the analysis. We begin with the impossibility of a no-draw equilibrium and then present a draw equilibrium with the above features.

Despite the presence of a minimum bid, Lemma 1 and its proof generalize virtually ad verbatim. Further, if an equilibrium were to occur in mixed strategies at a node n, it must involve both players bidding uniformly on $[m, \min\{\alpha_n, \beta_n\}]$ along with mass points at m and 0 by the higher and lower valued players respectively. The consequent net payoff for the lower valued player is zero while the higher valued player gets the entire rent $\max\{\alpha_n, \beta_n\} - \min\{\alpha_n, \beta_n\}$. In other words, Part 2 of Lemma 2 holds. Furthermore, the proof of Lemma 3 depends primarily[17] on this property and therefore, Part 2 of Lemma 3 and Corollary 1 readily generalize. Thus, a no-draw equilibrium, should one exist, must have $R = L+1$ or $R = L+3$ with $\alpha_n < \beta_n$ for $n \leq L$ and $\alpha_n \geq \beta_n$ for $n \geq R$.

Just as in the original then, there exists an integer L such that everywhere to its left, *Left* profits less than *Right* from winning a node and hence $u_n = \underline{u}_n$ and $\alpha_n = \underline{\alpha}_n \equiv -\left(1-\delta^2\right)\underline{u}_n$ still hold for such n. The utility computation for *Right* is now different. For, there could be a threshold integer $L_m < L$ where $\underline{\alpha}_{L_m} > m \geq \underline{\alpha}_{L_m+1}$ so that everywhere between $L_m + 1$ and L, only *Right* exerts a positive effort with the result $v_n = \delta v_{n-1} - m$. For $n \leq L_m$, as before, $v_n = \delta v_{n-1} - \underline{\alpha}_n$. Putting these together, we have a new expression for the utility of *Right* for $n \leq L$, i.e. $v_n = \hat{v}_n$ where

$$\hat{v}_n = \begin{cases} \overline{v}_n & \text{if } n \leq L_m \\ \delta^{n-L_m}\overline{v}_{L_m} - \left(1-\delta^{n-L_m}\right)\frac{m}{(1-\delta)} & \text{if } n > L_m \end{cases}$$

Finally, note that at a node n where only *Right* exerts a positive effort, her payoff if she chooses not to bid is δv_n which should not be greater than v_n her continuation payoff. Thus, $v_n \geq 0$ at all $n \leq L$ is required. If we let $\hat{n}_L$ denote the real number such that $\hat{v}_{\hat{n}_L} = 0$ and similarly define $\hat{n}_R$ with respect to the right end of the playing field, a no-draw equilibrium is an impossibility if

$$\hat{n}_L < \hat{n}_R \tag{20}$$

When does this inequality hold? It is easier to visualize the above condition in the limit as N diverges. Given m, note that the total minimum cost for transversing the entire playing field, for a given N, is $m\left(1-\delta^N\right)/(1-\delta)$. In taking the limit, we will ensure that the minimum bid is such that this total remains a constant. That is, pick a constant $M > 0$ and set $m \equiv m_N$ where

$$m_N = \frac{(1-\delta)}{(1-\delta^N)}M \tag{21}$$

[17] The reader would need only augment the proof of Lemma 3 with a few straightforward points that concern the behavior at a node where a pure strategy equilibrium in which exactly one player exerts positive effort (m).

It is easily seen that $\underline{u}_n$ converges pointwise to $\underline{u}(\lambda) = \mathrm{e}^{-\beta\lambda}u_0$ and $L_m/N \to \lambda_m$ where

$$-2\underline{u}(\lambda_m) = M/\left(1 - \mathrm{e}^{-\beta}\right).$$

With this in mind, and a little algebra, one can show that $\hat{v}_n$ converges pointwise to $\hat{v}(\lambda)$ where

$$\hat{v}(\lambda) = \begin{cases} \overline{v}(\lambda) & \text{if } \lambda \leq \lambda_m \\ \xi(\lambda)\,\overline{v}(\lambda_m) + (1 - \xi(\lambda))\frac{-M}{1-\mathrm{e}^{-}\beta} & \text{if } \lambda > \lambda_m \end{cases}$$

where $\xi(\lambda) = \mathrm{e}^{-\beta(\lambda - \lambda_m)}$.

Remark 1. Everywhere to the right of λ_m, note that $\hat{v}$ is a convex combination of a (typically) positive number $\overline{v}(\lambda_m)$ and a negative entity $-M/(1 - \mathrm{e}^{-}\beta)$. Further, a greater weight is added to the negative part at an exponential rate as one moves away to the right of λ. Moreover, λ_m moves inversely with M. These observations can be used to conclude that even for "moderate values" of M, $\hat{v}(1/2) < 0$. One can argue likewise for a symmetrically defined $\hat{u}(\cdot)$ that $\hat{u}(1/2) < 0$. Consequently, result (20) must obtain for high enough M.

We now turn to the existence of a draw equilibrium that has the properties described earlier on in this section. First define

$$\hat{\Psi}_n = \hat{v}_n + \underline{u}_n \tag{22}$$
$$\hat{\Phi}_n = \underline{v}_n + \hat{u}_n \tag{23}$$

where $\hat{u}_n$ is defined analogously to $\hat{v}_n$. L_m is as defined earlier but set L to be such that $\hat{\Psi}_L < 0 \leq \hat{\Psi}_{L-1}$. R_m is analogous to L_m while R satisfies $\hat{\Phi}_{R+1} \geq 0 > \hat{\Phi}_R$.

Remark 2. Suppose $R > L+2$. The following bidding behavior constitutes an equilibrium.

1. At $n = 1, \ldots, L_m$: *Right* bids m with probability m/α_n and according to the uniform distribution on $[m, \alpha_n]$ with the remaining probability. *Left* bids zero with probability $1 - (\alpha_n - m)/\beta_n$ and according to the uniform distribution on $[m, \alpha_n]$ with the remaining probability.

   Bidding at $n = R_m, \ldots, N-1$ is similar to the above.

2. At $n \in [L_m + 1, L - 2]$: *Right* bids m, *Left* does not bid. Likewise, at $n \in [R + 2, R_m - 1]$: *Left* bids m, *Right* does not bid.

3. At $n = L - 1$ and $n = R + 1$, either a pure strategy equilibrium as in (2) or a mixed strategy equilibrium as in (1) occurs.

4. At $n = L, \ldots, R$, neither player chooses a positive effort.

The table of the payoffs that correspond to the above equilibrium is presented as Remark 3 in the Appendix. This, together with steps similar to those in the proofs of Theorem 1 and Theorem 2 can be followed to verify that the above is indeed an equilibrium.

Just as its' counterpart with $m = 0$, note that in the above equilibrium too the sum of players' payoffs is u-shaped – and is given by max $\{\Psi_n, \Phi_n, 0\}$. Unless neither player exerts any effort, the maximum that is bid at any node is $\max\{\alpha_n, \beta_n, m\}$. Just as in the original, this is u-shaped.

The transition probabilities are rather different however. If at $n = L-1$ a pure strategy equilibrium were to occur, (See Part 3 above), it is impossible to transit from a region of fighting to a draw. This possibility does not occur when $m = 0$.

5.2 Other stage games

In this paper we have considered the CWA where a first price all pay auction is played at every node. The analysis of the model readily extends to the case where the stage game is some other standard auction (without reserve prices). In all such auctions, the player with the higher valuation wins and extracts a rent that equals the difference in the valuations. In other words, Lemma 2 can be seen to hold. Consequently, the same equilibrium payoffs as described by Theorem 1, Theorem 2 can be achieved.

There is a difference however in the probability of transition from one node to another. This can be illustrated by considering the Vickery Auction to be the stage game and payoffs as in a draw equilibrium. For all $n \leq L_0$, $\beta_n > \alpha_n$ and therefore *Right* wins such a node with probability one. Starting from such a node, *Right* will win the auction with probability one in n periods. At $n = L_0 + 1$, $\beta_n < \alpha_n$. Therefore, *Left* wins and with probability one, the transition to $n = L_0 + 2$, a state where neither player exerts any effort, occurs and the game ends in a draw. In contrast, there is a positive probability of win for *Right* at the $L_0 + 1$ in the CWA studied here which remains positive even in the limit at $N \to \infty$.

5.3 Budget Constraints

Although we have not explicitly modeled budget constrained players here, the CWA can be reinterpreted to accommodate them. Through a purification argument on the mixed strategies played at each node, our study can capture budgetary constraints on the *flow* of resources as follows.

Assume that at each date, the resource made available to a player[18] is the realization of a random variable distributed continuously according to $F(\cdot)$ on $[0, \overline{w}]$ for $\overline{w}$ sufficiently large. It may be useful to think of a player fighting based on the donations that she has received, arguably a reasonable description of Percy Schmeiser's affair with Monsanto.[19] Draws are independent across players and across nodes and observed privately. Also, wealth cannot be saved from one period to the next cannot.

The stage game is now a first price all pay auction with incomplete information about the other player's wealth but valuations α_n and β_n are common-knowledge. Che and Gale [1996] (see their Lemma 2) show that a Bayesian equilibrium of this game when $\alpha_n = \beta_n$ generates the same bid distribution as the mixed strategy of its complete information counterpart. It may be noted that their result does not depend in an essential way on $\alpha_n = \beta_n$. As a result, the entire analysis of CWA can be conducted with either one or two sided incomplete information about the budgetary flows of the opponents, under the assumption that resources cannot be stored, but can be spent on other things.

6 Conclusion

The present model accounts for interesting and salient features of the war of attrition. First, a lower cost of effort is an advantage. Second, there is what might be described as a momentum effect–as a player gets closer to winning, the player's likelihood of winning each battle, and the war, increases. Third, as a player gets closer to winning, and the other gets closer to losing, their efforts rise. Fourth, even in a model in which an infinitesimal effort can upset a tie, a draw is possible. Fifth, reducing a player's cost of effort will raise (lower) the expected conflict duration when that player is weaker (stronger) than his opponent. Sixth, at a node at the edge of a no-draw and draw region the likelihood of transition to draw is higher.

That a lower cost of effort leads to a greater likelihood of victory seems like a necessary condition for a war of attrition to be a plausible model of

[18] Could be either one or both players that are subject to this budget constraint.

[19] See Footnote 2 and http://www.percyschmeiser.com/ for details.

asymmetric contests. While weakness can be an advantage in some conflicts, it is generally only advantageous when weakness induces accommodation, as in the puppy dog strategy of Fudenberg and Tirole [1984]. As the term war of attrition is commonly understood, accommodation is not an option.

As one player gets close to winning, it seems quite reasonable that efforts rise. The winning player has an incentive to try to end the war quickly, since that is feasible. Similarly, for the losing player, increased effort delays defeat and the consequent utility loss. That effort is maximized near the endpoints of the game provides for "last ditch efforts" on the part of the (losing) defender, and an attempt to finish the job on the part of the (winning) offense. However, the result that total effort is u-shaped is at least a little surprising. In particular, since there is a discontinuity in the payoff around λ^*, one might have expected a good bit more effort devoted to pivoting around this point, rather than a passive acceptance of being on the losing side of λ^* on the part of the defense.

The existence of a draw is quite plausible, and appears to arise in actual conflicts, such as the Hundred Years War between France and Britain, which displayed long periods of a cessation of hostilities. In the Cold War between the United States and the Soviet Union, there was also a period of "peaceful coexistence," which could be interpreted as a draw. Theoretically, a draw should appear as a stationary equilibrium whenever it is too costly for one side to win when the other side devotes small levels of effort. Over the past 900 years, the position of Switzerland, relative to militarily strong neighbors, appears to fit this description. Switzerland had little value of winning a war against a neighbor, since it would be unlikely to succeed in extracting much surplus from a neighboring country. The militarily strong neighbor faced a difficult task to defeat Switzerland, because of the terrain (which creates a high cost of effort for an invading force), and, in this century, Switzerland's well-armed populace. As a consequence, the model appears to account for Swiss independence.

How can nations increase the likelihood of peaceful coexistence? The theory suggests that reducing the payoff to victory unambiguously increase the set of stable interior outcomes, which have a peaceful co-existence nature. Similarly, increasing the loss from defeat increases the set of peaceful coexistence nodes. These conclusions are reminiscent of the deterrence theory of warfare, which holds that deterrence arises when the balance of interests favors the defender[20]. In particular, it is the relative value of the defender

[20]See Achen and Snidel [1989] for an eloquent discussion of the theory and its relation to real situations. Schelling [1962] discusses the need for randomness in the outcome. As

and attacker that determines the outcome. The logic is that if the defender values the territory more than the attacker, the defender will have a stronger will to persist; in this event the attacker will lose. Attackers backward induct and decide not to attack in such circumstances. This is precisely the prediction of the model, in that peaceful co-existence occurs at all nodes left of λ^* when $u_0 + v_N < 0$, that is, the cost to the defender of losing exceeds the value to the attacker of winning[21].

Rational deterrence theory has been severely criticized on the grounds that conflicts occur when the theory suggests that the conflicts are not in either party's interests[22]. The present study suggests that multiple equilibria, one with deterrence or peaceful co-existence, one with war to the end, are a natural outcome in territorial disputes. The theory also suggests a distinction between strong deterrence, when peaceful co-existence is the unique equilibrium, and weak deterrence, when peaceful co-existence is one of two equilibria. Such a distinction may be useful in understanding the failures of rational deterrence theory[23].

A reduction of the cost of effort for one side has an ambiguous effect on peaceful co-existence. In the model, a reduction in the cost of effort for both by equal amounts, should reduce the scope for peaceful co-existence. This also seems plausible. In some sense, the gain from conflict has not changed, but its cost has been reduced, so the likelihood of conflict ought to increase. The model, therefore, can capture the idea that new weapons can be destabilizing even when held by both sides of the conflict. Weapons such as the neutron bomb are sometimes considered to be defensive only and not offensive. While the model does not readily incorporate the distinction between defensive and offensive weapons, the effects may be modeled by presuming that defensive weapons increase the cost of effort. Such an change increases the set of stable nodes in the model.

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commonly employed, the theory requires each country to be represented by a rational representative agent, and these agents playing a full information game.

[21] Lieberman [1995] uses the deterrence theory to account for the conflict between Egypt and Israel which followed shortly after the 6 day war, a conflict commonly called the War of Attrition (March, 1969-August 1970).

[22] See Achen and Snidel [1989] for a summary and critique.

[23] However, it is not sensible to insist on a full-information rational agent theory. Deterrence theory has much in common with limit pricing theory, and the approach taken by Milgrom and Roberts [1982] and offers significant insights for rational deterrence theory, including an understanding of bluffing (via pooling equilibria), and signalling (sabre-rattling). Some of the critics of rational deterrence theory are actually criticizing an implicit full information assumption. Given the secrecy employed by the military, the full information assumption is inappropriate.

When the value of winning is zero, all interior nodes are stable in the unique equilibrium. This no-win situation is the theory of mutually assured destruction for the model; by eliminating the value of winning, neither side has an incentive to fight, even when a player would like her opponent to lose to insure she doesn't lose.

The cost of effort has an ambiguous effect on the expected conflict duration. By making a strong (winning) player stronger, the player wins a larger proportion of battles and the war ends more quickly, while the reverse is true when the weak player is made stronger, unless the weak player is made sufficiently stronger to become the strong player.

How well does the model confront the colonial conflicts, such as Microsoft versus Netscape, Barnes and Noble versus Amazon.com, the US in Vietnam and Iraq, or the Union versus the Confederacy? In contrast to the standard model, being strong is an advantage. Moreover, a kind of momentum arises endogenously. Around the critical point λ^*, small gains can make significant differences in the likelihood of winning the conflict. Indeed, in the limiting continuum solution, the likelihood of winning is discontinuous at λ^*. In the case of Internet Explorer or Amazon.com, network externalities are sometimes identified as a reason that there will be an eventual winner. In the present model, the extreme form of network externalities (winner-take-all) imposed as a primitive translates into a critical point at which there is a discontinuity in payoffs.

In the model, as one side gets near to winning, both sides fight harder. In the military environment, the mortality rate for soldiers should rise near the end of the conflict. This seems implausible for many conflicts. In a business context, this prediction should be testable; advertising should be u-shaped in market share, and prices should be as well. As Internet Explorer's market share rose, the prices of both browsers fell, eventually to zero. The Department of Justice lawsuit against Microsoft probably confounds later observations about effort by the parties, for Microsoft was given a reason to accommodate Netscape's Navigator.

In colonial wars and market share fights, typically the holder of territory or market share derives a flow return roughly proportional to the market share. As a consequence, one might expect the contender in the lead to be able to devote more resources to the conflict, favoring that side. Such considerations appear to reinforce the instability of sharing the market.

Appendix

A First price war of attrition

Player 1 and Player 2 simultaneously choose how much to bid in a first price all pay auction with the following caveat: if the bids tie, Player i receives a gross benefit w_i. Otherwise, the higher bid wins, in which case the gross benefit of Player i is v_i if she wins and zero otherwise.

The above corresponds to the game played at each stage of the Continuing War of Attrition. The equilibrium of this game is presented below together with some of its relevant properties properties, but only for those parameter configurations which are relevant.

Proposition 4. *Suppose $v_i > w_i \geq 0$ for $i = 1, 2$ and let $v_1 \geq v_2$. There is a unique equilibrium, which is necessarily in mixed strategies. Player 1 bids according to the uniform distribution on $(0, v_2]$. Player 2 bids 0 with probability $1 - v_2/v_1$ and according to the uniform distribution on $(0, v_2]$ with the remaining probability. Moreover,*

1. *Equilibrium payoffs are $(v_1 - v_2, 0)$.*
2. *Player 1 wins with probability*
$$p = 1 - \frac{v_2}{2v_1}$$
3. *The maximum bid of either player is v_2.*
4. *Average bid of Player 1 is $v_2/2$ and that of Player 2 is $v_2^2/2v_1$.*
5. *Probability of a tie is zero.*

Proof. That the strategies described constitute an equilibrium can be easily verified as well as the characteristics described in Part 1-5 above. For a proof of uniqueness, follow the arguments that can be found in Hillman and Riley [1989] or Kovenock et al. [1996]. In these, proofs are offered for the case where $w_i = v_i/2$ but admit a direct extension to the case where $w_i < v_i$. □

B Proofs

Proof of Lemma 1. Suppose $u_n \leq 0$. If n is the penultimate node, the statement is true by assumption. Otherwise, let λ denote the probability of a tie at $n + 1$ if *Left* chooses not to bid. The resulting payoff is then $\lambda\delta u_{n+1} + (1-\lambda)\,\delta u_n \le u_{n+1}$, the payoff if she followed her equilibrium choice. Rearranging this inequality gives $(1-\lambda\delta)\,(u_{n+1}-u_n) \ge -(1-\delta)\,u_n$, which gives $u_{n+1} \ge u_n$, the inequality is strict whenever $u_n < 0$.

Now consider $u_n > 0$. The equilibrium at this node cannot be in pure strategies. Moreover, note that the closure of the supports of F_n and G_n must coincide, for otherwise there are values chosen with higher cost and no higher likelihood of advancing the node. Let x^* denote the supremum of *Right*'s bid distribution.

There are two cases to consider. First suppose $u_{n+1} \le u_{n-1}$. We claim that G_n and F_n are discrete distributions. If, by way of contradiction, some $x > 0$ is a point of continuity of G_n, bids in a left neighborhood $(x-\varepsilon, x)$ of x are dominated by bidding $x-\varepsilon$ for a small enough $\varepsilon > 0$, since in all these cases the probability of a tie is zero but the lower bid also lowers the probability of a disadvantageous advancement to the right. Such reasoning establishes that F_n and G_n must be discrete distributions with a common support. Consequently, x^* is actually in the support of F_n. Letting λ denote the probability of a tie when *Left* bids x^*, her payoff from that bid is $u_n = \delta\,(1-\lambda)\,u_{n+1} + \lambda\delta u_n - x^* < \delta\,(1-\lambda)\,u_{n+1} + \lambda\delta u_n$, which on simplification implies $u_n < u_{n+1}$.

Next, suppose $u_{n+1} > u_{n-1}$. It can be seen that the closure of the support of F_n (and G_n) is an interval. If x^* is a mass point of G_n with a mass of λ, then $u_n = \delta\,(1-\lambda)\,u_{n+1} - x^*$. On the other hand, by bidding an infinitesimal above x^*, here payoff would be $\delta u_{n+1} - x^*$. This should be bounded above by u_n but this is only possible if $u_{n+1} \le 0$, which in turn implies $u_n \le 0$, a contradiction. Therefore, x^* cannot be a mass point of either players bid distribution. Hence $u_n = \delta u_{n+1} - x^* < u_{n+1}$. □

Proof of Lemma 2. (Part 1) If the equalities hold, it is clear that it is an equilibrium for neither player to bid. To see the converse, if at all the players choose a pure action, it must be that both players bid zero, resulting in a tie. Consequently, $u_n = \delta u_n \Rightarrow u_n = 0$ and similarly $v_n = 0$. To eliminate *Left*'s incentive to bid even an infinitesimal more than zero, one must have $\delta u_{n+1} \le u_n$. From the first paragraph of Lemma 1, $u_{n+1} \ge u_n \Rightarrow u_{n+1} \ge 0$ and hence $u_{n+1} = 0$. Likewise, one must have $v_n = v_{n-1} = 0$.

(Part 2) The two players play a first price all pay auction in which *Left*'s value of a win, tie and a loss are α_n, α'_n and zero respectively. Likewise β_n, β'_n and zero for *Right*. By Lemma 1, $\alpha_n \ge \alpha'_n \ge 0$ and $\beta_n \ge \beta'_n \ge 0$. Part 1, Proposition 4 applies and the net payoffs are $\alpha_n - \min\{\alpha_n, \beta_n\}$ and $\beta_n - \min\{\alpha_n, \beta_n\}$ respectively. □

Proof of Lemma 3. The proof repeatedly relies on Lemma 1 and Lemma 2. *Part 1.* There are two cases to consider. Case 1 where a pure strategy equilibrium is played at n. Case 2 is where a mixed strategy equilibrium is played at n and $n+1$. Case 3 where the equilibrium is in mixed strategies at n but pure at $n+1$ and Case 4 which is symmetric to Case 3.

Case 1. Here If the equilibrium at n is in pure strategies, then $u_n = v_n = 0$ (Part 1, Lemma 2).

Case 2. From (2) and (3), $u_n = \delta u_{n+1} - \delta\left(v_{n-1} - v_{n+1}\right)$ and $v_n = \delta v_{n+1}$, $u_{n+1} = \delta u_n$ and therefore $u_n = \delta^2 u_n + v_n - \delta v_{n-1}$. On the other hand, if a pure strategy equilibrium is being played at $n-1$, then $u_n = 0$ which implies $v_n - \delta v_{n-1} = 0$, and by Lemma 1, $v_{n-1} \geq 0$. Therefore, $v_n \geq 0 = u_n$. Should the equilibrium at $n-1$ be in mixed strategies, then $v_{n-1} \geq \delta v_n$ and $u_n = \delta^2 u_n + v_n - \delta v_{n-1}$ reduces to $\left(1-\delta^2\right)\left(u_n - v_n\right) \leq 0$ and again $u_n \leq v_n$.

Case 3. Now suppose that the equilibria at n and $n+1$ are in mixed and pure strategies respectively. Then $u_{n+1} = v_{n+1} = v_{n-1} = 0$ which gives $u_n = \delta u_{n+1} + v_n - \delta v_{n-1} = -\delta v_{n-1}$. From Lemma 1, $v_{n-1} \geq v_n = 0$ and hence $u_n \leq 0 = v_n$. A symmetric argument shows that $v_{n+1} \leq u_{n+1}$.

Case 4. This is similar to the previous case.

Part 2. Suppose that $\beta_{n+2} \geq \alpha_{n+2}$ and assume, by way of contradiction that a mixed strategy equilibrium is played at $n+1$. Then, $u_{n+2} = \delta u_{n+1} = \delta^2 u_n$ which means $\alpha_{n+1} = -\delta\left(1-\delta^2\right) u_n$. Since $\alpha_{n+1} \geq 0$ must hold, it follows that $u_n \leq 0$. On the other hand, $v_{n+1} = \delta v_n - \alpha_{n+1}$ and since $v_n = \delta v_{n+1}$, we have $v_n = -\delta^2 u_n$. If $u_n < 0$ then $v_n > 0$ and contradicts Part 1 of the Lemma. Therefore, $u_n = 0$. But then $\alpha_{n+1} = 0$ and *Left* has no incentive to bid. Consequently, $v_{n+1} = \delta v_n - \alpha_{n+1} = \delta v_n = \delta^2 v_{n+1}$ which implies $v_{n+1} = 0$, $v_n = 0$, which is to say that player will have an incentive to exert any effort at $n+1$. The case when $\alpha_{n-1} \geq \beta_{n-1}$ is similar.

□

Proof of Theorem 1. Consider an equilibrium that involves a draw. Let ℓ be the smallest node at which a pure strategy equilibrium is played. Therefore, $u_\ell = u_{\ell+1} = v_\ell = v_{\ell-1} = 0$. Since a mixed strategy equilibrium occurs at $\ell-1$, $\beta_{\ell-1} = \delta\left(v_{\ell-2} - v_\ell\right) > 0$, i.e. $v_{\ell-2} > 0$. By Lemma 1, $v_n > 0$ for all $n \leq \ell-2$ and therefore $\beta_n > \alpha_n$ for all such n. Otherwise (3) cannot hold. Therefore, $u_n = \delta u_{n-1}$ for all $n \leq \ell-2$. Set $L = \ell-2$. Solving recursively, gives (10), except v_L. Arguing similarly for *Right* also gives us (11), except u_R. At $L+1 = \ell-1$, if it were the case that $\beta_{\ell-1} \geq \alpha_{\ell-1}$, then $u_{\ell-1} = \delta u_{\ell-2} < 0$ contradicting that $u_{\ell-1} = 0$. Therefore, it must be the case that $\alpha_{\ell-1} > \beta_{\ell-1}$ and consequently, $u_{\ell-1} = \delta\left(u_\ell + v_\ell\right) - \delta v_{\ell-2}$, i.e. $u_{L+1} = -\delta v_L$. Substitute for u_{L+1} in (10) to get $v_L = \delta\Psi_{L-1}/\left(1-\delta^2\right)$. To ensure $0 < v_{\ell-2} = v_L$, we require $\Psi_{L-1} > 0$. Also, to ensure $\alpha_{\ell-1} > \beta_{\ell-1}$, we must have $u_\ell + v_\ell > u_{\ell-2} + v_{\ell-2}$, i.e. $0 > u_L + v_L = \underline{u}_L + \delta\Psi_{L-1}/\left(1-\delta^2\right) = \Psi_L/\left(1-\delta^2\right)$. Therefore, $\Psi_L < 0$ is necessary. Therefore, $L = L_0$.

Moreover, from the above we have concluded that ℓ, the first node at which a pure strategy equilibrium is played equals $L_0 + 2$. Proceeding similarly, we conclude that $R = R_0$ and the last node at which a pure strategy equilibrium occurs, say ℓ' satisfies $\ell' = R_0 + 2$ where ℓ' is the last node at which a pure strategy equilibrium is played. Consequently, $R - L \geq 3$.

To see the converse, we need to verify that (4) and (5) hold. We will verify (4), the other is similar. First note that $L = L_0 < n_L$. From the comments that surround the definition of n_L, n_R in the text, we note that $s_{n-1} > s_{n+1}$ all $n \leq L-2$. It remains to verify them for $n = L-1, L, L+1$, i.e. $s_{L-2} > s_L$, $s_{L-1} > s_{L+1}$ and $s_L \leq s_{L+2}$. Each of these is immediate since, with the specified payoffs, (and recalling that a pure strategy equilibrium occurs at $n = L+2$)

$$\begin{aligned} s_L &= \frac{1}{1-\delta^2}\Psi_L \\ s_{L+1} &= -\frac{\delta^2}{1-\delta^2}\Psi_{L-1} \\ s_{L+2} &= 0 \end{aligned}$$

and $s_{L-2} = \Psi_{L-2}$, $s_{L-1} = \Psi_{L-1}$. □

Proof of Theorem 2. We know that in a no-draw equilibrium, either $R = L+3$ or $R = L+1$. As we already discussed when defining n_L and n_R, one must have $L-1 \leq n_L$ and $R+1 \geq n_R$. When $R = L+1$, the payoffs are given by (10) and (11) and (14). It remains to check that $s_{n-1} > s_{n+1}$ for $n \leq L$ if and $s_{n-1} \geq s_{n+1}$ for $n \geq R$ hold to ensure that can indeed be supported as equilibrium payoffs. For $n \leq L-2$ and $n \geq R+2$, these necessarily hold by the definition of n_L and n_R. It remains to check them remaining four inequalities: $s_{L-2} > s_L$, $s_{L-1} > s_{L+1}$, $s_L < s_{L+2}$ and $s_{L+1} < s_{L+3}$. The first two are

$$\underline{u}_L + v_L < \Psi_{L-2} \tag{24}$$

$$s_{L+1} < \Psi_{L-1} \tag{25}$$

Substituting for v_L we have the equivalent of (24):

$$\begin{aligned} \Psi_{L-2} &> \underline{u}_L + \frac{\delta}{1-\delta^2}\left(\Psi_{L-1} - \delta\Phi_{R+1}\right) \\ &= \frac{\Psi_L - \delta^2\Phi_{R+1}}{(1-\delta^2)} \end{aligned}$$

which simplifies to

$$\left(1-\delta^2\right)\Psi_{L-2}+\delta^2\Phi_{L+2}-\Psi_L \quad > \quad 0 \tag{26}$$

Substituting for s_{L+1} in (25) we obtain its equivalent

$$\begin{aligned}\Psi_{L-1} \quad &> \quad \frac{\delta}{1-\delta^2}\left(\Phi_{R+1}-\delta\Psi_{L-1}\right)+\underline{v}_{L+1} \\ &= \quad \frac{\Phi_R-\delta^2\Psi_{L-1}}{(1-\delta^2)}\end{aligned}$$

which simplies to

$$\Psi_{L-1} \quad > \quad \Phi_{L+1} \tag{27}$$

The analogous equivalent conditions corresponding to $s_L < s_{L+2}$ and $s_{L+1} < s_{L+3}$ are

$$\left(1-\delta^2\right)\Phi_{L+3}+\delta^2\Psi_{L-1}-\Phi_{L+1} \quad > \quad 0 \tag{28}$$

$$\Psi_L \quad < \quad \Phi_{L+2} \tag{29}$$

(26), (27), (28) and (29) along with $L-1 \le n_L$ and $R+1 \ge n_R$ constitute the necessary and sufficient conditions for a no-draw equilibrium in which $R=L+1$. However, on using (29),

$$\left(1-\delta^2\right)\Psi_{L-2}+\delta^2\Phi_{L+2}-\Psi_L \quad > \quad \left(1-\delta^2\right)\left(\Psi_{L-2}-\Psi_L\right)>0$$

Thus (29) $\Rightarrow$(26) and similarly (27) $\Rightarrow$(28). Consequently, the s necessary and sufficient condition for the existence of a no-draw equilibrium with $R = L+1$ is that $L < n_L, R < n_R$ and $\Psi_{L-1} > \Phi_{L+1}$ and $\Psi_L < \Phi_{L+2}$. The last two inequalities are satisfied only if L and $L+1$ lie on either side of the intersection Ψ_n of Φ_n, i.e. $n^* \in [L, L+1]$.

Implicit in the above arguments is the assumption that $L > 1$. The above analysis must be verified for the case where $L = 0$ or $L = N$. It is a routine computation to show that there is an equilibrium at $L = 0$ whenever $\Phi_{N-2} - \Psi_0 \ge 0$. Similarly $L = N$ arises when $\Psi_{N-2} - \Phi_0 \ge 0$. □

Proof of Proposition 3. It is convenient to introduce the following notation.

$$P_n = \prod_{i=1}^{n} \frac{p_i}{1-p_i}$$

From the expression for $(q_{n+1} - q_n)$ in Section **??**, we note that

$$\begin{aligned} q_{n+1} &= q_n + \frac{p_n}{1-p_n}\left(q_n - q_{n+1}\right) \\ &= q_1 \sum_{k=0}^{n} P_k \end{aligned}$$

and since $q_N = 1$ we have

$$q_n = \frac{\sum_{k=0}^{n} P_k}{\sum_{k=0}^{N-1} P_k} \tag{30}$$

Observe that for a bounded series $z_n = z\left(n/N\right)$,

$$\frac{\sum_{n=0}^{N-1} z_n P_n}{\sum_{n=0}^{N-1} P_n} \to z\left(\lambda^*\right) \tag{31}$$

(31) arises because $p_n > 0.5$ if, and only if, $n/N < \lambda^*$. Thus, as N gets large, an increasing weight of the probability mass in the expectation embodied in (31) is placed near $n = \lambda^* N$. Now $q\left(\lambda\right) = \lim_{n\to\infty} q_n$ is immediate by applying the above to (30).

Proceeding similarly to q_n, rewrite the difference equation for duration to give

$$\Delta_{n+1} - \Delta_n = \frac{p_n}{1-p_n}\left(\Delta_n - \Delta_{n-1}\right) - \frac{1}{1-p_n}$$

and solve recursively to get

$$\begin{aligned} \Delta_{n+1} - \Delta_n &= \Delta_1 \prod_{j=1}^{n} \frac{p_j}{1-p_j} - \sum_{j=1}^{n}\left(\frac{1}{1-p_j}\prod_{k=j+1}^{n}\frac{p_k}{1-p_k}\right) \\ &= P_n\left(\Delta_1 - \sum_{j=1}^{n}\frac{1}{\left(1-p_j\right)P_j}\right) \end{aligned}$$

which reduces to

$$\Delta_n = \sum_{m=1}^{n-1} P_m\left(\Delta_1 - \sum_{j=1}^{m}\frac{1}{\left(1-p_j\right)P_j}\right). \tag{32}$$

Moreover, substituting $\Delta_N = 0$ and solving for Δ_1 gives,

$$\Delta_1 \quad = \quad \frac{\sum_{k=1}^{N-1}\left(\sum_{j=1}^{n}\frac{1}{(1-p_j)P_j}\right)P_k}{\sum_{k=1}^{N-1}P_k}$$

Returning to (31), now observe that if $z_n = z\,(n/N) < 1-\varepsilon$ for some $\varepsilon > 0$,

$$\sum_{m=0}^{n}\left(\prod_{j=1}^{m} z_k\right) \to \frac{1}{1-z_0} \tag{33}$$

(33) follows from noting that, as N gets large, all of the relevant terms in the summation are very nearly powers of z 0. Formally, let z_a be the maximum of z over $[0,a]$, and $z_{\max}$ the maximum over [0,1]. Then

$$\sum_{m=0}^{n}\left(\prod_{j=1}^{m} z_k\right) = \sum_{m=0}^{aN}\left(\prod_{j=1}^{m} z_k\right) + \sum_{m=aN+1}^{\lambda N}\left(\prod_{j=1}^{m} z_k\right) \tag{34}$$

$$\leq \quad \frac{1-z_a^{aN+1}}{1-z_a} + \frac{z_a^{aN+1}}{1-z_{\max}} \to \frac{1}{1-z_a} \tag{35}$$

Sending $a \to 0$ completes the upper bound; the lower bound is analogous (using minima). Applying this to to the expression for Δ_1 above, we note that

$$\Delta_1 \quad \approx \quad \sum_{j=1}^{n^*}\frac{1}{(1-p_j)\,P_j}$$

for N sufficiently large, where $n^* = \lambda^* N$.

Thus, for $n < n^*$,

$$\begin{aligned}
\Delta_n \quad &\approx \quad \sum_{m=0}^{n-1}\left(\sum_{j=m+1}^{n^*}\frac{1}{(1-p_j)\,P_j}\right)P_m \\
&= \quad \sum_{m=0}^{n-1}\left(\sum_{j=m+1}^{n^*}\frac{1}{p_j}\prod_{k=1}^{j-1}\frac{1-p_k}{p_k}\right) \\
&\approx \quad \sum_{m=0}^{n-1}\frac{1}{p_m} \times \frac{1}{\left(1-\frac{(1-p_m)}{p_m}\right)} \\
&\approx \quad N\int_0^{n/N}\frac{1}{2p\,(x)-1}
\end{aligned}$$

The above establishes $\lim_{N\to\infty} \Delta_n/N$ when $n < n^*$. The case when $n > n^*$ is symmetric. □

Remark 3 (Concerning Remark 2)**.** The payoffs corresponding to the equilibrium described in Remark 2 above are given in the following table.

	$n \leq L-2$	$L-1$	$n = L, \ldots, R$	$R+1$	$n \geq R+2$
u_n	$\underline{u}_n$	$\underline{u}_{L-1}$	0	u_{R+1}	$\overline{u}_n$
v_n	$\overline{v}_n$	v_{L-1}	0	$\underline{v}_{R+1}$	$\underline{v}_n$

where

$$\begin{aligned} v_{L-1} &= \delta \overline{v}_{L-2} - \max\left\{-\underline{u}_{L-1}, m\right\} \\ u_{R+1} &= \delta \overline{u}_{R+2} - \max\left\{-\underline{v}_{R+1}, m\right\}. \end{aligned}$$

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