DISCRETE EQUILIBRIUM PRICE DISPERSION: EXTENSIONS AND TECHNICAL DETAILS

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1. Introduction

This paper provides supporting details and extensions to several results in Carlson and McAfee, "Discrete Equilibrium Price Dispersion," Journal of Political Economy, forthcoming. In what follows, a designation ((·)) refers to equations in the other paper, while a designation (·) refers to additional equations introduced here.

There are n firms. For each firm k, there are three key variables: p_k the price set by the firm, q_k the expected quantity demanded, and x_k the expected gain from further search for a buyer who has found price p_k. The initial model consists of the following three sets of n equations (for k=1,...,n):

$$x_k = \frac{1}{n} [(k-1)p_k - \sum_{i=2}^{k} p_i] \qquad ((3))$$

$$q_k = \frac{1}{n} G(x_{n+1}) - \frac{1}{k} G(x_k) + \sum_{j=k+1}^{n} \frac{1}{j(j-1)} G(x_j) \qquad ((5))$$

$$q_k + [p_k - c_k'(q_k)] \, \partial q_k/\partial p_k = 0 \qquad ((8))$$

where $c_k'(q_k)$ is the marginal cost of firm k and G(x) is a continuous function denoting the array of buyers as a function of the cost of search x. G(0) = 0 and $G(x_{n+1})$ denotes the total number of buyers.

Given specific functions for c' and G these equations can, in principle, be solved for each firm's price and quantity. This has been done for a quadratic cost function and a uniform distribution of search costs across customers and the solutions used to make comparative-statics predictions. Most of those predictions are stated in the other paper, but a number of details were omitted in the interest of simplifying the presentation. Here, we shall fill in those details.

In section 2, we show that equation ((3)) holds when customers do not necessarily have a precise knowledge of the distribution of actual prices in the market. In section 3, we derive a number of equations needed to obtain explicit solutions for the prices that the firms will set. Section 4 extends the analysis to study the effects of an increase in all customers' costs of search and to examine the stability of the firms' price-setting decisions.

2. Derivation of x_k with Imprecise Perceptions

The n prices have been ordered from the lowest p_1 to the highest p_n. Suppose consumers enter the market each period with the following perceived distribution of prices:

$$f(p) = \begin{cases} (1 - \delta/2)/n & p = p_1, p_n \\ (1 - \delta)/n & p = p_2, \ldots, p_{n-1} \\ \delta/n(p_i - p_{i-1} & p_{i-1} < p < p_i,\ i=2,\ldots,n \\ 0 & \text{elsewhere} \end{cases} \qquad (1)$$

with $0 \leq \delta \leq 1$. This distribution, pictured in Figure 1, has both discrete and continuous elements. This formulation of f(p) implicitly assumes that all prices are different. If two firms set the same price, then the probability mass at that price would become the sum of the probability mass of each separately plus δ/n. This, incidentally, is just one of many distributions that will yield the same demand functions.

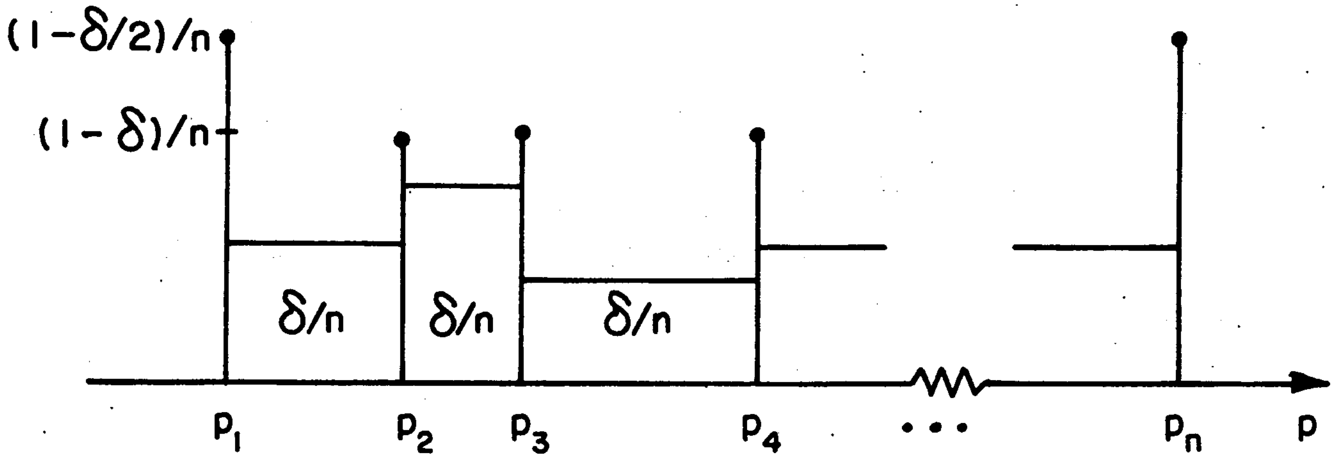

Figure 1

A Perceived Distribution of Prices

The parameter δ is a measure of uncertainty. If $\delta = 1$, f(p) is like a maximum entropy distribution used by Henri Theil, "Maximum Entropy Distributions: A Progress Report," Report 8043, University of Chicago, December 1980. As $\delta \to 0$, uncertainty disappears and the distribution becomes the true discrete distribution $f(p) = 1/n$. The mean of f(p) is:

$$\int_{p_1}^{p_n} pf(p)dp = \frac{(1-\delta)}{n} \sum_{i=1}^{n} p_i + \frac{\delta}{2n}(p_1+p_n) + \frac{\delta}{n} \sum_{i=2}^{n} \frac{1}{p_i-p_{i-1}} \int_{p_{i-1}}^{p_i} pdp$$

$$= \frac{1}{n} \sum_{i=1}^{n} p_i - \frac{\delta}{n} \sum_{i=1}^{n} p_i + \frac{\delta}{2n}[p_1+p_n + \sum_{i=2}^{n}(p_i+p_{i-1})] = \frac{1}{n} \sum_{i=1}^{n} p_i$$

Thus, we have allowed the consumers to know the highest price, the lowest price, and the mean price, and have some idea of prices in between without necessarily being certain about them.

Now consider the expected benefit x_k from continued search for a consumer who has encountered price p_k. In that case:

$$x_k = \int_o^{p_k} (p_k-p)f(p)dp = \frac{1-\delta/2}{n}(p_k-p_1) + \sum_{i=2}^{k} \frac{(1-\delta)}{n}(p_k-p_i)$$

$$+ \frac{\delta}{n} \sum_{i=2}^{k} \int_{p_{i-1}}^{p_i} \frac{p_k-p}{p_i-p_{i-1}} dp$$

$$= \frac{1-\delta}{n} \sum_{i=1}^{k} (p_k-p_i) + \frac{\delta}{2n}(p_k-p_1) + \frac{\delta}{n} \sum_{k=2}^{k} \frac{p_k p-p^2/2}{p_i-p_{i-1}} \Big|_{p_{i-1}}^{p_i}$$

$$= \frac{k(1-\delta)}{n} p_k - \frac{1-\delta}{n} \sum_{i=1}^{k} p_i + \frac{\delta}{2n}(p_k-p_1)$$

$$+ \frac{\delta}{n}[(k-1)p_k - \frac{1}{2} \sum_{i=2}^{k} (p_i+p_{i-1})]$$

$$= \frac{1}{n}[(k-1)p_k - \sum_{i=2}^{k} p_i]$$

This is the same as ((3)).

3. Supporting Details for the Explicit Equilibrium Solution

To get closed-form expressions for the demand functions ((5)), we make use of the following:

LEMMA: $$\sum_{i=0}^{k} \frac{1}{(n+i)(n+i-1)} = \frac{k+1}{(n-1)(n+k)} .$$

PROOF: By induction. Clearly true for k=0.

If it is true for k, then

$$\sum_{i=0}^{k+1} \frac{1}{(n+i)(n+i-1)} = \frac{k+1}{(n-1)(n+k)} + \frac{1}{(n+k+1)(n+k)}$$

$$= \frac{(n+k+1)(k+1)+(n-1)}{(n-1)(n+k)(n+k+1)} = \frac{(n+k)(k+2)}{(n-1)(n+k)(n+k+1)}$$

$$= \frac{k+2}{(n-1)(n+k+1)} .$$

In particular, in terms of the expressions in ((5)), the Lemma can be used to show that:

$$\sum_{k=j+1}^{n} \frac{1}{k(k-1)} = \frac{n-j}{nj} . \qquad (2)$$

This relationship will be used in deriving equation ((12)) below.

Let the search costs be distributed across consumers such that

$$\begin{aligned} G(x) &= x/s \qquad && 0 \le x \le T \\ &= T/s && T < x \end{aligned} \qquad ((11))$$

If this is substituted into ((5)), noting that $x_{n+1} \geq T$:

$$q_j = \frac{T}{sn} - \frac{1}{sj} x_j + \sum_{k=j+1}^{n} \frac{x_k}{k(k-1)s}$$

Then, substituting from ((3)):

$$q_j = \frac{1}{s} [\frac{T}{n} - \frac{1}{j} [\frac{j-1}{n} p_j - \frac{1}{n} \sum_{i=1}^{j-1} p_i] + \frac{1}{n} \sum_{k=j+1}^{n} \frac{(k-1)p_k - \sum_{i=1}^{k-1} p_i}{k(k-1)}]$$

$$= \frac{1}{sn} [T - \frac{j-1}{j} p_j + \frac{1}{j} \sum_{i=1}^{j-1} p_i + \sum_{i=j+1}^{n} \frac{p_i}{i} - \sum_{k=j+1}^{n} \sum_{i=1}^{k-1} \frac{p_i}{k(k-1)}]$$

$$= \frac{1}{sn} [T - \left(\frac{j-1}{j} + \sum_{k=j+1}^{n} \frac{1}{k(k-1)}\right) p_j + \sum_{i=1}^{j-1} p_i[\frac{1}{j} - \sum_{k=j+1}^{n} \frac{1}{k(k-1)}]$$

$$+ \sum_{i=j+1}^{n} p_i[\frac{1}{i} - \sum_{k=i+1}^{n} \frac{1}{k(k-1)}]$$

$$= \frac{1}{sn} [T - \left(\frac{j-1}{j} + \frac{n-j}{nj}\right) p_j + \sum_{i=1}^{j-1} p_i[\frac{1}{j} - \frac{n-j}{nj}] + \sum_{i=j+1}^{n} p_i[\frac{1}{i} - \frac{n-i}{ni}]]$$

$$= \frac{1}{sn} [T - \frac{n-1}{n} p_j + \sum_{i=1}^{j-1} \frac{1}{n} p_i + \sum_{i=j+1}^{n} \frac{1}{n} p_i]$$

$$= \frac{1}{sn} [T - p_j + \sum_{i=1}^{n} \frac{p_i}{n}] = \frac{1}{sn} [T - \frac{n-1}{n} p_j + \sum_{i \neq j} \frac{p_i}{n}] \qquad (3)$$

or, equivalently

$$q_j = \frac{1}{sn} [T - (p_j - \bar{p})] \qquad ((12))$$

where

$$\bar{p} = \sum_{j=1}^{n} p_j/n$$

This is the case in which a firm's demand is a linear function of the difference between its price and the average price in the market. Also, note from (3) that

$$\frac{\partial q_j}{\partial p_j} = - \frac{n-1}{sn^2} \tag{4}$$

With quadratic costs, a firm's profit function is:

$$\pi_j = (p_j - \alpha_j - \beta q_j)q_j \qquad ((15))$$

The first-order condition for maximum profit $\partial\pi_j/\partial p_j = 0$ can be written:

$$\frac{\partial \pi_j}{\partial p_j} = q_j + (p_j - \alpha_j - 2\beta q_j)\frac{\partial q_j}{\partial p_j} \tag{5}$$

$$= \frac{1}{sn}\left(T - \frac{n-1}{n}p_j + \sum_{i \neq j}\frac{p_i}{n}\right) + (p_j - \alpha_j - 2\beta q_j)\left(-\frac{n-1}{sn^2}\right)$$

$$= \frac{1}{sn}\left(T - \frac{n-1}{n}p_j + \sum_{i \neq j}\frac{p_i}{n}\right)\left(1 + \frac{2\beta(n-1)}{sn^2}\right) - (p_j - \alpha_j)\left(\frac{n-1}{sn^2}\right) = 0$$

Define $\gamma = \frac{2\beta(n-1)}{sn^2}$ and multiply (5) through by sn^2.

$$(nT - (n-1)p_j + \sum_{i \neq j} p_i)(1+\gamma) - (p_j - \alpha_j)(n-1) = 0$$

Thus,

$$(n-1)(2+\gamma)p_j = nT(1+\gamma) + (n-1)\alpha_j + \sum_{i \neq j}(1+\gamma)p_i \qquad ((16))$$

To solve for equilibrium prices, note that for j=i

$$(n-1)(2+\gamma)p_i = nT(1+\gamma) + (n-1)\alpha_i + \sum_{k \neq i}(1+\gamma)p_k \tag{6}$$

Subtract (6) from ((16)) and collect terms

$$(2n-1+\gamma n)(p_j-p_i) = (n-1)(\alpha_j-\alpha_i)$$

or

$$p_i = p_j + \frac{n-1}{2n-1+\gamma n}(\alpha_j-\alpha_i) \tag{7}$$

Sum these for $i \neq j$, substitute for $\sum_{i \neq j} p_i$ in ((16)) and collect terms:

$$(n-1)p_j = nT(1+\gamma) + (n-1)\alpha_j + \sum_{i \neq j} \frac{(1+\gamma)(n-1)}{2n-1+\gamma n}(\alpha_i-\alpha_j)$$

which can be rewritten

$$p_j = \alpha_j + \frac{nT(1+\gamma)}{n-1} + \sum_{i=1}^{n} \frac{(1+\gamma)}{2n-1+\gamma n}(\alpha_i-\alpha_j)$$

This can be put into the form:

$$p_j = \alpha_j + \frac{(1+\gamma)n}{n-1}\left[T + \frac{n-1}{2n-1+\gamma n}(\bar{\alpha}-\alpha_j)\right] \tag{(18)}$$

To show the proportionality of profits to quantity squared, from ((12)) and ((18)):

$$q_j = \frac{1}{sn}\left[T + \frac{n-1}{2n-1+\gamma n}(\bar{\alpha}-\alpha_j)\right]$$

From this and from ((18)):

$$p_j - \alpha_j - \beta q_j = \left[T + \frac{n-1}{2n-1+\gamma n}(\bar{\alpha}-\alpha_j)\right]\left[\frac{(1+\gamma)n}{n-1} - \frac{\beta}{sn}\right]$$

Therefore, $\pi_j = (p_j-\alpha_j-\beta q_j)q_j$ can be seen to be proportional to q_j^2.

4. Extensions

We next consider the case in which the domain of the distribution of search costs is shifted to the right:

$$G(x) = \begin{cases} 0 & 0 \le x < w \\ (x-w)/s & w \le x \le T + w \\ T/s & T + w < x \end{cases} \tag{8}$$

A technical problem arises because the demand specifications vary depending on how many firms charge the lowest price. To illustrate, suppose only firm 1 charges the lowest price. Then from ((5)):

$$q_1 = \frac{1}{sn}\,(T - (n-1)w - p_1 + \bar{p})$$

$$q_j = \frac{1}{sn}\,(T + w - p_j + \bar{p}) \qquad j=2,\ldots,n$$

If precisely two firms charge the lowest price, then

$$q_j = \frac{1}{sn}\,(T - (\tfrac{n}{2} - 1)w - p_j + \bar{p}) \qquad j=1,2$$

$$q_j = \frac{1}{sn}\,(T + w - p_j + \bar{p}) \qquad j=3,\ldots,n$$

This shift in search costs removes a group of consumers who will search out the lowest price and in effect adds that demand to all the other firms. The difference is in how many lowest priced firms share this loss in business. Sinc there is little to be gained in keeping track of all the cases, we shall present the results when a single firm charges the lowest price, that is, when $\alpha_1 < \alpha_2$.

The first-order conditions for each firm's profit maximization yield:

$$\frac{\partial \pi_j}{\partial p_j} = q_j + (p_j - \alpha_j - 2\beta q_j)\partial q_j/\partial p_j = \qquad (\text{for } j \neq 1)$$

$$\frac{1}{sn}\,(T + w - p_j + \bar{p}) + (p_j - \alpha_j - 2\beta q_j)\,\left(-\frac{n-1}{sn^2}\right) = 0$$

Multiply through by sn^2 and collect terms in p_j:

$$(n-1)(2+\gamma)p_j = n(T+w)(1+\gamma) + (n-1)\alpha_j + \sum_{i\neq j} (1+\gamma)p_i \quad j\neq 1 \tag{9}$$

A similar procedure for firm 1 yields:

$$(n-1)(2+\gamma)p_1 = n(T-(n-1)w)(1+\gamma) + (n-1)\alpha_1 + \sum_{i=2}^{n} (1+\gamma)p_i \tag{10}$$

(9) and (10) reduce to ((16)) when $w = 0$ and describe the price that each firm would set given the prices set by other firms. Subtracting (10) from (9), collecting terms, and solving for p_j:

$$p_j = p_1 + \frac{n-1}{2n-1+\gamma n} (\alpha_j-\alpha_1) + \frac{n^2(1+\gamma)w}{2n-1+\gamma n} \tag{11}$$

Substituting this into (10):

$$(n-1)(2+\gamma)p_1 = n(1+\gamma)T - n(n-1)(1+\gamma)w + (n-1)\alpha_1$$

$$+ \sum_{i\neq 1} (1+\gamma)[p_1 + \frac{n-1}{2n-1+\gamma n} (\alpha_i-\alpha_1) + \frac{n^2(1+\gamma)w}{2n-1+\gamma n}]$$

So

$$(n-1)p_1 = n(1+\gamma)T + n(n-1)(1+\gamma)[\frac{n(1+\gamma)-2n+1-\gamma n}{2n-1+\gamma n}] w$$

$$+ (n-1)\alpha_1 + \sum_{i=1}^{n} \frac{(n-1)(1+\gamma)}{2n-1+\gamma n} (\alpha_i-\alpha_1)$$

Hence

$$p_1 = \frac{n(1+\gamma)T}{n-1} - \frac{n(1+\gamma)(n-1)}{2n-1+\gamma n} w + \alpha_1 + \frac{n(1+\gamma)}{2n-1+\gamma n} (\bar{\alpha} - \alpha_1) \tag{12}$$

(11) and (12) can then be used to get the equilibrium prices:

$$p_j^* = \alpha_j + \frac{n(1+\gamma)}{n-1} [T + \frac{n-1}{2n-1+\gamma n} (\bar{\alpha} - \alpha_j + w)] \qquad j=2,\ldots,n \tag{13}$$

$$p_1^* = \alpha_1 + \frac{n(1+\gamma)}{n-1} [T + \frac{n-1}{2n-1+\gamma n} (\bar{\alpha} - \alpha_1 - (n-1)w)] \tag{14}$$

A seemingly peculiar result for this equilibrium solution is that

$$\frac{\partial p_1^*}{\partial w} = -\frac{n(n-1)(1+\gamma)}{2n-1+\gamma n} < 0$$

$$\frac{\partial p_j^*}{\partial w} = \frac{n(1+\gamma)}{2n-1+\gamma n} > 0 \qquad \text{for } j\neq 1$$

If consumers' search costs all rise by the same amount, the lowest cost firm lowers its price and all other firms raise their prices. The explanation is that the lowest-priced firm loses some of its natural competitive advantage in terms of customers who will search until they find p_1. Firm 1 can recapture some of this business by lowering p_1 and increasing $x_2, \ldots, x_n$ (the consumer's expected benefits from further search after encountering $p_2, \ldots, p_n$, respectivel[y]). Firm 2 also responds to not losing as much business to firm 1 by raising p_2, which also helps increase x_2. All other firms raise their prices by the same amount as firm 2, so the differences $x_k - x_{k-1}$, $(k = 3, \ldots, n)$ are unchanged and the x_k $(k = 2, \ldots, n)$ are all increased by the same amount.

It is readily established from (13) and (14) that

$$\bar{p} = \bar{\alpha} + \frac{(1+\gamma)nT}{n-1}$$

Therefore $\partial\bar{p}/\partial w = 0$. The mean equilibrium price is not changed. It can also be shown that var p is increased. Thus, a tax on searching or any uniform increase in search costs tends to increase the variance of prices while leaving the mean price unchanged.

If all search costs are positive, there may or may not be an equilibrium price distribution. Clearly, there must be some customers for whom the benefit from searching out the lowest price exceeds their cost of search in order for there to be some advantage in being the lowest priced firm. Thus, we need $w \leq x_2 = (p_2 - p_1)/n$. This can be shown to require $w \leq (\alpha_2-\alpha_1)/n$. From (13) and (14):

$$\frac{p_2-p_1}{n} = \frac{1}{n}\left[\alpha_2-\alpha_1 + \frac{n(1+\gamma)}{2n-1+\gamma n}\ (-\alpha_2+\alpha_1 + nw)\right]$$

$$= \frac{(n-1)(\alpha_2-\alpha_1) + n^2(1+\gamma)w}{n(2n-1+\gamma n)} \leq w$$

$\Leftrightarrow$

$$\frac{(n-1)(\alpha_2-\alpha_1)}{n} \leq (2n-1+\gamma n-n-\gamma n)w$$

$\Leftrightarrow$

$$\frac{\alpha_2-\alpha_1}{n} \leq w$$

As $n \to \infty$, the upper bound on w can be made arbitrarily small. Thus with a continuum of firms, the result of no equilibrium solution when w is strictly positive and consumers engage in sequential search can be seen to hold.

Equations (9) and (10) may be thought of as reaction functions. Given the prices set by other firms, firm j can maximize its profits by setting its price according to these functions. We can prove that $w < (\alpha_2-\alpha_1)/n$ is a least upper bound for which the price distribution does not break down by showing that not only is the equilibrium price vector a unique local profit maximization for every firm but also is globally stable. To see this, note that (9) and (10) can be written:

$$(n-1)(2+\gamma)p_j = z_j + \sum_{k\neq j} (1+\gamma)p_k \qquad (15)$$

Suppose $p_1^*,\ldots,p_n^*$ satisfy (15), suppose initial prices are $p_1^* + \varepsilon_1$, $\ldots$, $p_n^* + \varepsilon_n$, and let the adjustment in each price be denoted δ_j. Then

$$(n-1)(2+\gamma)(p_j^* + \varepsilon_j + \delta_j) = z_j + \sum_{k\neq j} (1+\gamma)(p_k^* + \varepsilon_k)$$

By (15) and the definition of $p_1^*, \ldots, p_n^*$,

$$\varepsilon_j + \delta_j = \frac{(1+\gamma)}{(n-1)(2+\gamma)} \sum_{k\neq j} \varepsilon_k$$

So

$$\sum_{j=1}^{n} (\varepsilon_j + \delta_j)^2 = \sum_{j=1}^{n} [\frac{(1+\gamma)}{(n-1)(2+\gamma)} (\sum_{k=1}^{n} \varepsilon_k - \varepsilon_j)]^2 =$$

$$\frac{(1+\gamma)^2}{(n-1)^2(2+\gamma)^2} \sum_{j=1}^{n} [\varepsilon_j^2 + (\sum_{k=1}^{n} \varepsilon_k)^2 - 2\varepsilon_j \sum_{k=1}^{n} \varepsilon_k] =$$

$$\frac{(1+\gamma)^2}{(2+\gamma)^2(n-1)^2} [\sum_{j=1}^{n} \varepsilon_j^2 + (n-2)(\sum_{k=1}^{n} \varepsilon_k)^2] \leq$$

by Cauchy-Schwarz inequality

$$\frac{(1+\gamma)^2}{(2+\gamma)^2(n-1)^2} [\sum_{j=1}^{n} \varepsilon_j^2 (1+(n-2)n)] = (\frac{1+\gamma}{2+\gamma})^2 \Sigma\varepsilon_j^2 < \Sigma \varepsilon_j^2$$

since we have $\gamma \geq 0$. Thus, the system is stable. From any starting point it corrects via a contraction mapping. See Hans Sagan, Advanced Calculus, Houghton Mifflin, 1974, p. 263. The speed of convergence is less than or equal to $(\frac{1+\gamma}{2+\gamma})^2 < 1$.