Auctioning Entry into Tournaments

TECHNICAL APPENDIX
(Not Intended For Publication)

(This corresponds to a slightly earlier version than the published version, so you may have to account for slight differences.)

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Technical Appendix

(Expanded proofs, not intended for publication)

The following lemma is used repeatedly in the paper. This lemma was first proven in the generality shown here by Guesnerie and Laffont (1984); however, special cases were used by several authors, notably Myerson (1981) prior to this. Subscripts denote partial derivatives.

Lemma A1: *Suppose v:* $[a,b] \to \Re$ *is twice continuously differentiable. Then*

$$(\forall r)(\forall x)\ \ v(r,x) \le v(x,x) \ \text{ implies} \qquad \text{(A1)}$$
$$(\forall x)\ \ v_1(x,x) = 0 \ \text{ and} \qquad \text{(A2)}$$
$$(\forall x)\ \ v_{12}(x,x) \ge 0.\ \text{ Moreover, (A2) and} \qquad \text{(A3)}$$
$$(\forall r)(\forall x)\ \ v_{12}(r,x) \ge 0 \ \text{ imply (A1).} \qquad \text{(A4)}$$

Section II

There are $i = 1, 2, \ldots, n$ firms that desire entry. The cost of research for each of these firms is c_i and they must decide after entry how much effort, z_i, to conduct so the total cost of firm i's research is $c_i z_i$ along with a fixed cost of γ. Given effort z_i the CDF of firm i's best innovation is $F^{z_i}(x)$ with a density of $z_i F^{z_i - 1}(x) f(x)$. If there are J firms that participate and each firm starts with an initially worthless innovation the expected profit, excluding the fixed costs of γ, for each participant is:

$$(1) \qquad \pi_i = P\int_{\underline{x}}^{\overline{x}} \left[\prod_{j \ne i} F^{z_j}(x) \right] z_i F^{z_i - 1}(x) f(x)\,dx - c_i z_i = \frac{P z_i}{\sum_j z_j} - c_i z_i$$

and (2)

$$\frac{\partial \pi_i}{\partial z_i} = \frac{P \sum_{j \ne i} z_j}{\left(\sum_j z_j \right)^2} - c_i \qquad \text{and} \qquad \frac{\partial^2 \pi_i}{(\partial z_i)^2} = \frac{-2P \sum_{j \ne i} z_j}{\left(\sum_j z_j \right)^3} - c_i < 0$$

THEOREM 1: Given a set of firms, there is a unique equilibrium in the subgame, which involves positive z_i for the lowest cost firms.

Proof: Order the set of firms from lowest cost to highest cost, for the purpose of this proof only subscripts will refer to the ordered costs within the set of firms. An equilibrium is a subset of these firms M such that $z_i = 0$ for $i \notin$ M and $z_i > 0$ for $i \in$ M. Therefore,

$$i \notin \mathrm{M} \;\Rightarrow\; 0 \geq \left.\frac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} = \frac{P}{\sum_{j \in M} z_j} - c_i \quad \text{or} \quad c_i \geq \frac{P}{\sum_{j \in M} z_j}$$

$$i \in \mathrm{M} \;\Rightarrow\; c_i = \frac{P \sum_{\substack{j \in M \\ j \neq i}} z_j}{\left(\sum_{j \in M} z_j\right)^2} \quad \text{and} \quad z_i > 0$$

Let $|M|$ refer to the number of elements in M. Summing the first order condition over $i \in$ M, we have:

$$\sum_{i \in M} c_i = \frac{P(|M|-1)\sum_{j \in M} z_j}{\left(\sum_{j \in M} z_j\right)^2} = \frac{P(|M|-1)}{\sum_{j \in M} z_j} \quad \text{or} \quad \sum_{j \in M} z_j = \frac{P(|M|-1)}{\sum_{i \in M} c_i}$$

The requirement that $z_i > 0$ yields:

$$\text{For } i \in \mathrm{M}\text{:}\;\; c_i = \frac{P \sum_{\substack{j \in M \\ j \neq i}} z_j}{\left(\sum_{j \in M} z_j\right)^2} < \frac{P \sum_{j \in M} z_j}{\left(\sum_{j \in M} z_j\right)^2} = \frac{\sum_{j \in M} c_j}{(|M|-1)} \qquad \text{For } i \notin \mathrm{M}\text{:}\;\; c_i \geq \frac{P}{\sum_{j \in M} z_j} = \frac{\sum_{j \in M} c_j}{(|M|-1)}$$

Therefore, any equilibrium involves only the lowest cost firms, say m of them, and is characterized by:

$$c_m < \frac{\sum_{j=1}^{m} c_j}{m-1} \leq c_{m+1}$$

This is unique because: $\frac{\sum_{j=1}^{K} c_j}{K-1} \leq c_{K-1} \quad \Rightarrow \quad \sum_{j=1}^{K} c_j \leq (K-1)c_{K-1} \quad \Rightarrow \quad \frac{\sum_{j=1}^{K+1} c_j}{K} \leq c_{K+1} \leq c_{K+2}$

Thus, by induction: $\frac{\sum_{j=1}^{m} c_j}{m-1} \leq c_{m+1} \quad \Rightarrow \quad \forall K \geq m, \quad \frac{\sum_{j=1}^{K} c_j}{K-1} \leq c_{K+1}$ **Q.E.D.**

As part of Theorem 1, we obtain the expressions (3) and (4) when M is the set of firms that choose positive z:

From the term $\sum_{j \in M} z_j = \frac{P(|M|-1)}{\sum_{i \in M} c_i}$ rearrange to get $z_i = \frac{P(|M|-1)}{\sum_{i \in M} c_i} - \sum_{\substack{j \in M \\ j \neq i}} z_j$ now substitute to get:

(3) $$z_i = \frac{P(|M|-1)}{\sum_{i \in M} c_i} - \frac{c_i}{P}\left(\sum_{j \in M} z_j\right)^2 = \frac{P(|M|-1)}{\sum_{j \in M} c_j}\left[1 - \frac{c_i(|M|-1)}{\sum_{j \in M} c_j}\right]$$

From $\pi_i = \frac{P z_i}{\sum_{j \in M} z_j} - c_i z_i$ rearrange $\pi_i = z_i\left[\frac{P}{\sum_{j \in M} z_j} - c_i\right] = \frac{P(|M|-1)}{\sum_{j \in M} c_j}\left[1 - \frac{c_i(|M|-1)}{\sum_{j \in M} c_j}\right]\left(\frac{P}{\sum_{j \in M} z_j} - c_i\right)$

and substitute to get (4): $$\pi_i = \frac{P(|M|-1)}{\sum_{j \in M} c_j}\left[1 - \frac{c_i(|M|-1)}{\sum_{j \in M} c_j}\right]\left(\frac{\sum_{j \in M} c_j}{(|M|-1)} - c_i\right) = P\left[1 - \frac{c_i(|M|-1)}{\sum_{j \in M} c_j}\right]^2$$

LEMMA 1: Suppose M represents an entry equilibrium, and $k \in$ M, $i \notin$ M. Let $m = |M|$. Then,

$$c_i \geq \frac{m^2 - m}{m^2 - m + 1} c_k$$

<u>Proof:</u> Suppose $c_i < c_k = \max_{j \in M}\{c_j\}$, and $i \notin M$.

Since $k \in$ M, $P\left[1 - \frac{c_k\left(|M|-1\right)}{\sum_{j \in M} c_j}\right]^2 \geq \gamma$, or $\frac{c_k\left(|M|-1\right)}{\sum_{j \in M} c_j} \leq 1 - \sqrt{\gamma / P}$

If $c_k \geq \frac{c_i + \sum_{j \in M} c_j}{|M|}$, then entry of i would cause $z_k = 0$, yielding profit for firm i of:

$$\pi_i = P\left[1 - \frac{c_i\left(|M|-1\right)}{\sum_{j \in M} c_j - c_k + c_i}\right]^2 > P\left[1 - \frac{c_k\left(|M|-1\right)}{\sum_{j \in M} c_j}\right]^2 \geq \gamma$$

but this implies $i \in$ M which contradicts our earlier supposition. Therefore, $c_k < \frac{c_i + \sum_{j \in M} c_j}{|M|}$.

Since $i \notin$ M: $P\left[1 - \frac{c_i|M|}{c_i + \sum_{j \in M} c_j}\right]^2 \leq \gamma$, or $\frac{c_i|M|}{c_i + \sum_{j \in M} c_j} \geq 1 - \sqrt{\gamma / P}$. Letting $m = |M|$, we have:

$$\frac{c_i m}{c_i + \sum_{j \in M} c_j} \geq \frac{c_k(m-1)}{\sum_{j \in M} c_j}, \quad \text{or} \quad c_i m \geq c_k(m-1)\left[1 + \frac{c_i}{\sum_{j \in M} c_j}\right] \geq c_k(m-1)\left[1 + \frac{c_i}{m c_k}\right]$$

where the last inequality holds because $c_k = \max_{j \in M}\{c_j\}$.

Multiplying through by mc_k to get: $m^2 c_i c_k \geq c_k(m-1)\left[mc_k + c_i\right]$ or $[m^2 - (m-1)]c_i \geq m(m-1)c_k$

which is the result stated. **Q.E.D.**

THEOREM 2: There is a unique number firms, *m*, which is efficient for entry in the tournament and entry of the *m* lowest-cost firms is an equilibrium.

Proof: Note that $\gamma \geq P\left[1-\frac{c_k(k-1)}{\sum_{j\leq k} c_j}\right]^2$ iff $1-\sqrt{\gamma/P} \leq \frac{c_k(k-1)}{\sum_{j\leq k} c_j}$

implies: $\left(1-\sqrt{\gamma/P}\right)\sum_{j\leq k} c_j \leq c_k(k-1) \Rightarrow \left(1-\sqrt{\gamma/P}\right)\sum_{j\leq k+1} c_j \leq c_k(k-1)+\left(1-\sqrt{\gamma/P}\right)c_{k+1} \leq kc_{k+1}$

implies that: $\gamma \geq P\left[1-\frac{kc_{k+1}}{\sum_{j\leq k+1} c_j}\right]^2$. Thus, if it is unprofitable for firm *k* to enter, it is unprofitable for firm *k+1* to enter, showing the equilibrium *m* is uniquely determined. **Q.E.D.**

Define: $\Delta_m = \frac{mc_m}{\sum_{j=1}^{m} c_j}$ where c_m is the cost of the m^{th} lowest cost firm.

THEOREM 3: If Δ_m is nondecreasing, then the total procurement cost of obtaining a fixed level Z is minimized at *m* = 2.

Proof: $Z = \sum_{j=1}^{m} z_j = \frac{P(m-1)}{\sum_{j=1}^{m} c_j} \Rightarrow P = \frac{Z\sum_{j=1}^{m} c_j}{m-1}$

The optimal entry fee with *m* participants is: $E = P\left[1-\frac{c_m(m-1)}{\sum_{j\in M} c_j}\right]^2 - \gamma$

which are the profits of the m^{th} highest firm. The total cost of procurement with *m* participants is:

$$TC_m = P - mE = P\left\{1-m\left[1-\frac{c_m(m-1)}{\sum_{j\in M} c_j}\right]^2\right\}+m\gamma = \frac{Z\sum_{j=1}^{m} c_j}{m-1}\left\{1-m\left[1-\frac{c_m(m-1)}{\sum_{j\in M} c_j}\right]^2\right\}+m\gamma$$

$$= Z\sum_{j=1}^{m} c_j \left\{ -1 + \frac{2mc_m}{\sum_{j=1}^{m} c_j} - \frac{m(m-1)c_m^2}{\left(\sum_{j=1}^{m} c_j\right)^2} \right\} + m\gamma = Z\sum_{j=1}^{m} c_j \left\{ -1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2 \right\} + m\gamma$$

Note that $1 \le \Delta_m \le \frac{m}{m-1}$, since $\frac{(m-1)c_m}{\sum_{j=1}^{m} c_j} < 1$ (profits are nonnegative) and $c_j \le c_m,\ \forall\, j < m$.

$$TC_{m+1} - TC_m \;=\; \gamma + Z\sum_{j=1}^{m+1} c_j \left\{ -1 + 2\Delta_{m+1} - \frac{m}{m+1}\Delta_{m+1}^2 \right\} - Z\sum_{j=1}^{m} c_j \left\{ -1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2 \right\}$$

$$= \gamma + Z\sum_{j=1}^{m+1} c_j \left\{ -1 + 2\Delta_{m+1} - \frac{m}{m+1}\Delta_{m+1}^2 \right\} - Z\sum_{j=1}^{m+1} c_j \left\{ -1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2 \right\}$$

$$+ Zc_{m+1}\left\{ -1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2 \right\}$$

$$= \gamma + Z\sum_{j=1}^{m+1} c_j \left\langle \left\{ 2\Delta_{m+1} - \frac{m}{m+1}\Delta_{m+1}^2 - 2\Delta_m + \frac{m-1}{m}\Delta_m^2 \right\} + \frac{\Delta_{m+1}}{m+1}\left\{ -1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2 \right\} \right\rangle$$

$$= \gamma + Z\sum_{j=1}^{m+1} c_j \left\langle 2(\Delta_{m+1} - \Delta_m) - \frac{m}{m+1}\Delta_{m+1}^2 + \frac{m-1}{m}\Delta_m^2 + \frac{\Delta_{m+1}}{m+1}\left\{ -1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2 \right\} \right\rangle$$

$$= \gamma + Z\sum_{j=1}^{m+1} c_j \left\langle 2(\Delta_{m+1} - \Delta_m) - (\Delta_{m+1} - \Delta_m)\left(\frac{m}{m+1}\Delta_{m+1} + \frac{m-1}{m}\Delta_m \right) - \frac{\Delta_m \Delta_{m+1}}{m(m+1)} \right.$$

$$\left. + \frac{\Delta_{m+1}}{m+1}\left\{ -1 + 2\Delta_m - \frac{m-1}{m}\Delta_m^2 \right\} \right\rangle$$

$$= \gamma + Z\sum_{j=1}^{m+1} c_j \left\langle (\Delta_{m+1} - \Delta_m)\left(2 - \frac{m}{m+1}\Delta_{m+1} - \frac{m-1}{m}\Delta_m \right) + \frac{\Delta_{m+1}}{m+1}\left\{ -1 + \frac{\Delta_m(2m-1)}{m} - \frac{m-1}{m}\Delta_m^2 \right\} \right\rangle$$

$$= \gamma + Z\sum_{j=1}^{m+1} c_j \left\langle (\Delta_{m+1} - \Delta_m)\left(1 - \frac{m}{m+1}\Delta_{m+1} + 1 - \frac{m-1}{m}\Delta_m \right) - \frac{\Delta_{m+1}}{m+1}(1 - \Delta_m)\left\{ 1 - \frac{m-1}{m}\Delta_m \right\} \right\rangle$$

↑ positive by hypothesis

↑ positive since terms like: $1 - \frac{m-1}{m}\Delta_m \ \ge\ 0$

↑ negative

↑ positive

Therefore, this expression is clearly positive as long as $\Delta_{m+1} - \Delta_m \geq 0$, implying $TC_{m+1} \geq TC_m$ and the optimum occurs at *m* = 0. **Q.E.D.**

LEMMA 2: Suppose $m \leq \overline{m}$ and $\frac{c_m}{c_{m+1}} \leq \frac{1}{m} + \frac{m-1}{m}\frac{c_{m-1}}{c_m}$. Then $\Delta_m \leq \Delta_{m+1}$.

__**Proof:**__ (By induction) First note $c_2 \geq c_1 \Rightarrow \Delta_2 \geq \Delta_1$. Rewrite $\Delta_m \geq \Delta_{m-1}$ as: $1 - \frac{(m-1)c_{m-1}}{mc_m} \geq \frac{c_m}{\sum_{j=1}^{m} c_j}$

By assumption, $\frac{1}{m} + \frac{m-1}{m}\frac{c_{m-1}}{c_m} \geq \frac{c_m}{c_{m+1}}$. Therefore $1 - \frac{m-1}{m}\frac{c_{m-1}}{c_m} \geq \frac{c_m}{\sum_{j=1}^{m} c_j}$ implies:

$\frac{m+1}{m} = 1 - \frac{(m-1)c_{m-1}}{mc_m} + \frac{1}{m} + \frac{(m-1)c_{m-1}}{mc_m} \geq \frac{c_m}{\sum_{j=1}^{m} c_j} + \frac{c_m}{c_{m+1}}$ which implies $\Delta_{m+1} \geq \Delta_m$. **Q.E.D.**

Section III:

In an *m* contestant auction, firm *i's* profits are: $\pi_i = P\left[1 - \frac{c_i(m-1)}{\sum_{j=1}^{m} c_j}\right]^2$

Consider the *m*+1st (uniform) price auction, and suppose *B* is an equilibrium bidding function. In order for the equilibrium to be efficient, *B* must be decreasing in higher costs. A firm with cost c_i which bids $B(\hat{c})$ earns profits:

$$\pi_i(\hat{c}, c_i) = P\int_{\hat{c}}^{\infty}\left\{\int_{o}^{c_m}\cdots\int_{o}^{c_m}\left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]^2 \frac{h(c_1)}{H(c_m)}\frac{h(c_2)}{H(c_m)}\cdots\frac{h(c_{m-1})}{H(c_m)}dc_1 dc_2 \ldots dc_{m-1}\right\}h_{m:n-1}(c_m)dc_m$$

$$-\int_{\hat{c}}^{\infty} B(c_m)h_{m:n-1}(c_m)dc_m$$

(NOTE: In this problem, we are looking for the *m* lowest cost firms, therefore we use standard order statistic notation where $h_{m:n-1}(x)$ refers to the *m*th lowest cost firm out of (*n*-1) firms.)

Taking the partial of this profit with respect to $\hat{c}$ gives us:

$$\frac{\partial \pi_i}{\partial \hat{c}} = -P\left\{\int_o^{\hat{c}} \ldots \int_o^{\hat{c}} \left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]^2 \frac{h(c_1)}{H(\hat{c})}\frac{h(c_2)}{H(\hat{c})} \ldots \frac{h(c_{m-1})}{H(\hat{c})} dc_1 dc_2 \ldots dc_{m-1}\right\} h_{m:n-1}(\hat{c}) + B(\hat{c}) h_{m:n-1}(\hat{c})$$

Since $\frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}$ is increasing in c_i this implies that $\frac{\partial^2 \pi_i}{\partial \hat{c} \partial c_i} \geq 0$. Thus, using Theorem A1 by Guesnerie and Laffont, in equilibrium:

(6) $$B(c_i) = P\int_o^{c_i} \ldots \int_o^{c_i} \left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]^2 \frac{h(c_1)}{H(c_i)}\frac{h(c_2)}{H(c_i)} \ldots \frac{h(c_{m-1})}{H(c_i)} dc_1 dc_2 \ldots dc_{m-1}$$

If this equilibrium bidding function *B(c)* is decreasing in *c*, then the *m*+1st price auction is efficient in selecting the lowest cost firms for entry in the tournament.

Lemma 3: If $\frac{ch(c)}{H(c)}$ is decreasing in *c*, then *B(c)* is decreasing in *c*. Alternately, if $\frac{ch(c)}{H(c)}$ is nondecreasing in *c*, then *B(c)* is nondecreasing.

Proof:

$$\frac{\partial B(c_i)}{\partial c_i} = P(m-1)\int_o^{c_i} \ldots \int_o^{c_i} \left[1 - \frac{c_i(m-1)}{2c_i + \sum_{j=1}^{m-2} c_j}\right]^2 \frac{h(c_1)}{H(c_i)}\frac{h(c_2)}{H(c_i)} \ldots \frac{h(c_{m-2})}{H(c_i)} dc_1 dc_2 \ldots dc_{m-2} \frac{h(c_i)}{H(c_i)}$$

$$-P(m-1)\frac{h(c_i)}{H(c_i)}\int_o^{c_i} \ldots \int_o^{c_i} \left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]^2 \frac{h(c_1)}{H(c_i)}\frac{h(c_2)}{H(c_i)} \ldots \frac{h(c_{m-1})}{H(c_i)} dc_1 dc_2 \ldots dc_{m-1}$$

$$-P\int_o^{c_i} \ldots \int_o^{c_i} 2\left[1 - \frac{c_i(m-1)}{c_i + \sum_{j=1}^{m-1} c_j}\right]\left\{\frac{(m-1)\sum_{j=1}^{m-1} c_j}{\left[c_i + \sum_{j=1}^{m-1} c_j\right]^2}\right\} \frac{h(c_1)}{H(c_i)}\frac{h(c_2)}{H(c_i)} \ldots \frac{h(c_{m-1})}{H(c_i)} dc_1 dc_2 \ldots dc_{m-1}$$

We can integrate the second term by parts using $\int u dv = uv - \int v du$ and letting:

$$u=\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]^2 \quad \text{so} \quad du=2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left(\frac{c_i(m-1)}{\left\{c_i+\sum_{j=1}^{m-1}c_j\right\}^2}\right)dc_{m-1}$$

$$dv=\frac{h(c_1)h(c_2)...h(c_{m-1})}{\left[H(c_i)\right]^{m-1}}dc_{m-1} \quad \text{so} \quad v=\frac{h(c_1)h(c_2)...h(c_{m-2})}{\left[H(c_i)\right]^{m-1}}H(c_{m-1})$$

Therefore, the second term can be rewritten:

$$-P(m-1)\frac{h(c_i)}{H(c_i)}\left\langle \int_o^{c_i}...\int_o^{c_i}\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]^2\frac{h(c_1)h(c_2)...h(c_{m-2})}{\left[H(c_i)\right]^{m-1}}H(c_{m-1})dc_1dc_2...dc_{m-2}\Bigg|_0^{c_i}\right.$$

$$\left.-\int_0^{c_i}...\int_0^{c_i}2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{c_i(m-1)}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\frac{h(c_1)h(c_2)...h(c_{m-2})}{\left[H(c_i)\right]^{m-1}}H(c_{m-1})dc_1dc_2...dc_{m-1}\right\rangle$$

Substituting this expression in for the second term we get:

$$\frac{\partial B(c_i)}{\partial c_i}=P(m-1)\int_o^{c_i}...\int_o^{c_i}\left[1-\frac{c_i(m-1)}{2c_i+\sum_{j=1}^{m-2}c_j}\right]^2\frac{h(c_1)h(c_2)...h(c_{m-2})}{\left[H(c_i)\right]^{m-1}}dc_1dc_2...dc_{m-2}h(c_i)$$

$$-P(m-1)\frac{h(c_i)}{H(c_i)}\left\langle \int_o^{c_i}...\int_o^{c_i}\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]^2\frac{h(c_1)h(c_2)...h(c_{m-2})}{\left[H(c_i)\right]^{m-1}}H(c_{m-1})dc_1dc_2...dc_{m-2}\Bigg|_0^{c_i}\right.$$

$$\left.-\int_0^{c_i}...\int_0^{c_i}2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{c_i(m-1)}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\frac{h(c_1)h(c_2)...h(c_{m-2})}{\left[H(c_i)\right]^{m-1}}H(c_{m-1})dc_1dc_2...dc_{m-1}\right\rangle$$

$$-P\int_o^{c_i}...\int_o^{c_i}2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{(m-1)\sum_{j=1}^{m-1}c_j}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\frac{h(c_1)h(c_2)...h(c_{m-1})}{\left[H(c_i)\right]^{m-1}}dc_1dc_2...dc_{m-1}$$

But the first two terms cancel out, and we are left with:

$$\frac{\partial B(c_i)}{\partial c_i} = P(m-1)\frac{h(c_i)}{H(c_i)}\int_0^{c_i}\ldots\int_0^{c_i} 2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]$$

$$\times\left\{\frac{c_i(m-1)}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\frac{h(c_1)h(c_2)\ldots h(c_{m-2})}{\left[H(c_i)\right]^{m-1}}H(c_{m-1})dc_1dc_2\ldots dc_{m-1}$$

$$-P\int_o^{c_i}\ldots\int_o^{c_i} 2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{(m-1)\sum_{j=1}^{m-1}c_j}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\frac{h(c_1)h(c_2)\ldots h(c_{m-1})}{\left[H(c_i)\right]^{m-1}}dc_1dc_2\ldots dc_{m-1}$$

Rearranging terms:

$$\frac{\partial B(c_i)}{\partial c_i} = P\frac{(m-1)}{\left[H(c_i)\right]^m}\left\langle h(c_i)(m-1)\int_0^{c_i}\ldots\int_0^{c_i} 2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\right.$$

$$\times\left\{\frac{c_i}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}h(c_1)h(c_2)\ldots h(c_{m-2})H(c_{m-1})dc_1dc_2\ldots dc_{m-1}$$

$$\left.-\int_o^{c_i}\ldots\int_o^{c_i} 2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{\left[H(c_i)\right]\sum_{j=1}^{m-1}c_j}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}h(c_1)h(c_2)\ldots h(c_{m-1})dc_1dc_2\ldots dc_{m-1}\right\rangle$$

Putting into summation notation: $\frac{\partial B(c_i)}{\partial c_i} =$

$$P\frac{(m-1)}{\left[H(c_i)\right]^m}\left\langle\sum_{k=1}^{m-1}h(c_i)\int_0^{c_i}\ldots\int_0^{c_i} 2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{c_iH(c_k)}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\prod_{j\neq k}^{m-1}h(c_j)dc_1\ldots dc_{m-1}\right.$$

$$\left.-\sum_{k=1}^{m-1}\int_o^{c_i}\ldots\int_o^{c_i} 2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{\left[H(c_i)\right]c_k}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\prod_{j=1}^{m-1}h(c_j)dc_1\ldots dc_{m-1}\right\rangle$$

Combining terms:

$$\frac{\partial B(c_i)}{\partial c_i}=P\frac{(m-1)}{\left[H(c_i)\right]^m}\sum_{k=1}^{m-1}\int_0^{c_i}\cdots\int_0^{c_i}2\left[1-\frac{c_i(m-1)}{c_i+\sum_{j=1}^{m-1}c_j}\right]\left\{\frac{1}{\left[c_i+\sum_{j=1}^{m-1}c_j\right]^2}\right\}$$

$$\times\prod_{j\neq k}^{m-1}h(c_j)\left[c_ih(c_i)H(c_k)-c_kh(c_k)H(c_i)\right]dc_1\ldots dc_{m-1}$$

Rearranging and dropping the *i* subscripts we are left with:

$$\frac{\partial B(c)}{\partial c}=P\frac{(m-1)}{\left[H(c)\right]^{m-1}}\sum_{k=1}^{m-1}\int_0^{c}\cdots\int_0^{c}2\left[1-\frac{c(m-1)}{c+\sum_{j=1}^{m-1}c_j}\right]$$

$$\times\left\{\frac{H(c_k)}{\left[c+\sum_{j=1}^{m-1}c_j\right]^2}\right\}\prod_{j\neq k}^{m-1}h(c_j)\left[\frac{ch(c)}{H(c)}-\frac{c_kh(c_k)}{H(c_k)}\right]dc_1\ldots dc_{m-1}$$

Therefore, if $\frac{ch(c)}{H(c)}$ is *decreasing* in *c*, then since $c_k < c_i$ this implies $\left[\frac{ch(c)}{H(c)}-\frac{c_kh(c_k)}{H(c_k)}\right]<0$ and $\frac{\partial B(c)}{\partial c}<0$ so a pure strategy equilibrium exists that is efficient with the lowest cost firms submitting the highest auction bids. On the other hand if $\frac{ch(c)}{H(c)}$ is increasing, then any equilibrium is inefficient which was the conclusion of Section III.

Section IV.

First we will show there is no differentiable pure-strategy bidding equilibrium for the uniform-price auction when contestants differ in starting values and have no opportunity for further research following entry. If we define *"y"* to be the cutoff starting value of the (M+1)st firm, the probability firm *i* wins the prize, given entry is our equation (7). A firm holding starting innovation x_i that bids as if it held innovovation *r* would expect a profit equal to:

(8) $$\pi(r, x_i) = P \int_0^{\min\{r, x_i\}} \left(\frac{F(x_i) - F(y)}{1 - F(y)} \right)^{m-1} f_{m:n}(y)dy - \int_0^{r} B(y) f_{m:n}(y)dy$$

The first order conditions for a maximum require $\pi_1(x_i, x_i) = 0$.

$$\pi_1(r, x_i) = P\left(\frac{F(x_i) - F(r)}{1 - F(r)} \right)^{m-1} f_{m:n}(r) - B(r) f_{m:n}(r) \qquad \text{if } r < x_i$$

$$\pi_1(r, x_i) = -B(r) f_{m:n}(r) \qquad \text{if } r > x_i$$

In both cases, evaluating at r = x_i gives us: $\pi_1(x_i, x_i) = -B(x_i) f_{M:N}(x_i)$

Setting $\pi_1(x_i, x_i) = 0$ implies the only possible equilibrium bidding function is $B(x) = 0 \quad \forall x$.

Which clearly cannot be an equilibrium bidding function.

Now consider the case where firms can conduct additional research following entry. In what follows we will prove a series of lemmas to derive the equations and conditions which characterize the profits of these firms as represented by [Graph 1] in the paper.

Let x_i be firm i's initial innovation endowment, and $x_{max} = \max_i \{x_i\}$.

For $x_i < x_{max}$, firm i wins the tournament with probability:

(9) $$\int_{x_{\max}}^{\infty} \prod_{j \neq i} F(y)^{z_j} z_i F(y)^{z_i - 1} f(y)dy = \frac{z_i}{\sum_j z_j} \left[1 - F(x_{\max})^{\sum_j z_j} \right]$$

In contrast, the firm holding x_{max} wins the tournament with probability:

(10) $$F(x_{\max})^{\sum_j z_j} + \frac{z_i}{\sum_j z_j} \left[1 - F(x_{\max})^{\sum_j z_j} \right]$$

Lemma 4: If the value of the best entrant's starting innovation is such that

$F(x_{\max}) \geq e^{-c/P}$ **then none of the tournament contestants will conduct additional research following entry.**

Proof: The choice of z under full information and identical costs is solved as follows:

(For simplicity define $Z \equiv \sum_{j=1}^{m} z_j$)

For $i \neq max$: $\pi_i = P\frac{z_i}{Z}\left[1 - F(x_{\max})^Z\right] - cz_i$

$$\frac{\partial \pi_i}{\partial z_i} = \frac{P(Z - z_i)}{Z^2}\left[1 - F(x_{\max})^Z\right] - \frac{Pz_i}{Z}F(x_{\max})^Z \log F(x_{\max}) - c \quad \& \quad \left.\frac{\partial \pi_i}{\partial z_i}\right|_{z_i = 0} = P\frac{1 - F(x_{\max})^Z}{Z} - c$$

$$\frac{1}{P}\frac{\partial^2 \pi_i}{(\partial z_i)^2} = -2\frac{(Z - z_i)}{Z^3}\left[1 - F(x_{\max})^Z\right] - 2\frac{(Z - z_i)}{Z^2}F(x_{\max})^Z \log F(x_{\max}) - \frac{z_i}{Z}F(x_{\max})^Z[\log F(x_{\max})]^2$$

$$\leq -2\frac{(Z - z_i)}{Z^3}\left[1 - F(x_{\max})^Z + ZF(x_{\max})^Z \log F(x_{\max})\right] \leq 0$$

To see that the term in the square brackets is positive, note that it comes in the form:

$\varphi(Z) = 1 - F^Z + ZF^Z \log F$ where $\varphi(0) = 0$ and $\varphi'(Z) = ZF^Z (\log F)^2 \geq 0.$

For $i = max$: $\pi_{\max} = PF(x_{\max})^Z + P\frac{z_{\max}}{Z}\left[1 - F(x_{\max})^Z\right] - cz_{\max}$

$$\frac{\partial \pi_{\max}}{\partial z_{\max}} = P\left[1 - \frac{z_{\max}}{Z}\right]F(x_{\max})^Z \log F(x_{\max}) + \frac{P(Z - z_{\max})}{Z^2}\left[1 - F(x_{\max})^Z\right] - c$$

$$= P\frac{Z - z_{\max}}{Z^2}\left[ZF(x_{\max})^Z \log F(x_{\max}) + 1 - F(x_{\max})^Z\right] - c$$

$$\operatorname{sgn} \frac{\partial^2 \pi_{\max}}{(\partial z_{\max})^2} = \operatorname{sgn} \frac{\partial}{\partial Z}\frac{\left[ZF(x_{\max})^Z \log F(x_{\max}) + 1 - F(x_{\max})^Z\right]}{Z^2}$$

$$= \operatorname{sgn} \frac{Z^2 F(x_{\max})^Z\left[\log F(x_{\max})\right]^2 - 2\left[ZF(x_{\max})^Z \log F(x_{\max}) + 1 - F(x_{\max})^Z\right]}{Z^3}$$

$$= \operatorname{sgn} Z^2\left[\log F(x_{\max})\right]^2 - 2Z \log F(x_{\max}) - 2\left[F(x_{\max})^{-Z} - 1\right]$$

To sign this last expression, let $\varphi(Z) = Z^2[\log F]^2 - 2Z\log F - 2[F^{-Z} - 1]$ so that $\varphi(0) = 0$ and

$\varphi'(Z) = 2Z[\log F]^2 - 2\log F + 2F^{-Z}\log F$ therefore $\varphi'(0) = 0$ and

$\varphi''(Z) = 2[\log F]^2 - 2F^{-Z}[\log F]^2 = 2[\log F]^2\{1 - F^{-Z}\} < 0$

Therefore, $\varphi \le 0$ which implies $\dfrac{\partial^2 \pi_{max}}{(\partial z_{max})^2} \le 0$.

$$\left.\frac{\partial \pi_{max}}{\partial z_{max}}\right|_{z_m=0} = \frac{P}{Z}\left[1 - F(x_{max})^Z + ZF(x_{max})^Z \log F(x_{max})\right] - c$$

and previously we showed $\left.\dfrac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} = \dfrac{P}{Z}\left[1 - F(x_{max})^Z\right] - c$ therefore $\left.\dfrac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} \ge \left.\dfrac{\partial \pi_{max}}{\partial z_{max}}\right|_{z_{max}=0}$

So, $\left.\dfrac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} \le 0 \quad \Rightarrow \quad \left.\dfrac{\partial \pi_{max}}{\partial z_{max}}\right|_{z_{max}=0} \le 0$.

Thus, Z = 0 IFF $\left.\dfrac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} \le 0$ IFF $\lim\limits_{Z\to 0} \dfrac{P}{Z}\left[1 - F(x_{max})^Z\right] - c \le 0$

Using L'Hopital's rule this is: $\lim\limits_{Z\to 0} -PF(x_{max})^Z \log[F(x_{max})] - c \le 0$

IFF $\log[F(x_{max})] \ge -\dfrac{c}{P}$ IFF $[F(x_{max})] \ge e^{-c/P}$

Thus, Z = 0 IFF $F(x_{max}) \ge e^{-c/P}$ which is the result we sought to prove.

Q.E.D.

Now we will derive the equations and conditions which characterize the profits of these firms as represented by [Graph 1] in the paper.

Define $\bar{u}_m$ to be the solution to: $1 - u + mu\log u$ in (0,1). This solution is unique.

To see this, let: $\psi(u) = 1 - u + mu\log u.$ $\psi(0) = 1$ and $\psi(1) = 0$.

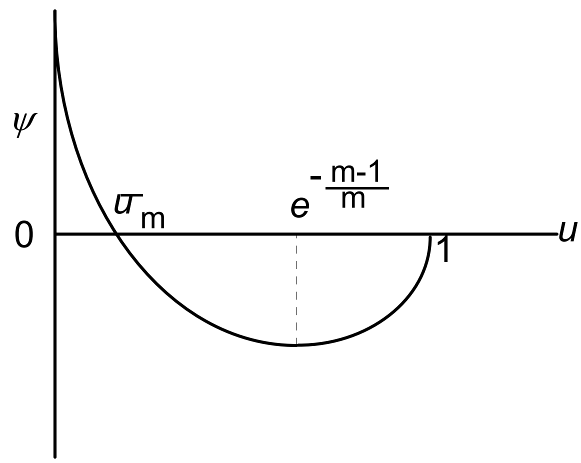
$$\psi'(u) = -1 + u + m[1 + \log u] = m = 1 + m \log u \;\overset{\le}{\underset{\ge}{}}\; 0 \quad \text{as} \quad u \;\overset{\le}{\underset{\ge}{}}\; e^{-(m-1)/m}$$

This is an interior solution because $\psi'(1) = m - 1 > 0$. Note that $\bar{u}_m < \frac{1}{m}$, because for $m \ge 2$,

$$\psi\left(\tfrac{1}{m}\right) = 1 - \frac{1}{m} + \log\frac{1}{m} = \frac{m-1}{m} - \log(m) < 0$$

Example:

m	***(m-1)*** $\bar{u}_m$	***m***	***(m-1)*** $\bar{u}_m$
2	0.284668	10	0.242264
3	0.297999	20	0.208155
4	0.289949	100	0.152669
5	0.279681	1000	0.109550



In fact, *(m-1)* $\bar{u}_m \to 0$ as $m \to \infty$. Let $c_m =$ *(m-1)* $\bar{u}_m$. Then $1 - \frac{c_m}{m-1} + \frac{m c_m}{m-1}\log\left(\frac{c_m}{m-1}\right) = 0.$

For large *m* since $c_m \le \frac{m-1}{m}$, then $1 + c_m \log\left(\frac{c_m}{m-1}\right) \approx 0.$ Therefore, if $c_m \to c_\infty > 0$ then

$c_m \log\left(\frac{c_m}{m-1}\right) \to -\infty$, which is a contradiction.

Lemma A2: If $-\log F(x_{\max}) < \frac{c}{P}$, $Z = 0$

If $-(m-1)\bar{u}_m \log F(x_{\max}) \le \frac{c}{P} \le -\log F(x_{\max})$, $z_{\max} = 0$ and Z solves:

$$\frac{m-2}{m-1}\;\frac{1 - F^Z(x_{\max})}{Z} - \frac{1}{(m-1)} F^Z(x_{\max}) \log F(x_{\max}) - \frac{c}{P} = 0$$

If $\frac{c}{P} < -(m-1)\bar{u}_m \log F(x_{\max})$, $z_{\max} = 0$ and Z solves $\frac{m-1}{m}\;\frac{1 - F^Z(x_{\max})}{Z} - \frac{c}{P} = 0$

Proof: The first portion of this lemma was already shown in Lemma 4 above.

For $i \neq max$, $\frac{\partial \pi_i}{\partial z_i} = P\frac{(Z-z_i)[1-F^Z(x_{\max})]}{Z^2} - \frac{Pz_i F^Z(x_{\max})}{Z}\log F(x_{\max}) - c$

$$\frac{\partial \pi_{\max}}{\partial z_{\max}} = P\frac{(Z-z_{\max})[1-F^Z(x_{\max})]}{Z^2} + \frac{P(Z-z_{\max})F^Z(x_{\max})}{Z}\log F(x_{\max}) - c$$

First note that, If $i, j \neq max$, then $z_i = z_j$.

For $\left.\frac{\partial \pi_j}{\partial z_j}\right|_{z_j=0} = \frac{P[1-F^Z(x_{\max})]}{Z^2} - c = \frac{\partial \pi_i}{\partial z_i} + \frac{Pz_i}{Z}\left[\frac{[1-F^Z(x_{\max})}{Z} + F^Z(x_{\max})\log F(x_{\max})\right]$

Thus, if $z_i > 0$, $z_j > 0$ as $\frac{1-F^Z(x_{\max})}{Z} + F^Z(x_{\max})\log F(x_{\max}) > 0.$

But then, $0 = \frac{\partial \pi_i}{\partial z_i} - \frac{\partial \pi_j}{\partial z_j} = P\frac{(z_j - z_i)}{Z}\left[\frac{1-F^Z(x_{\max})}{Z} - F^Z(x_{\max})\log F(x_{\max})\right]$, which implies $z_i = z_j$.

However, $\left.\frac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} \leq 0 \Rightarrow \left.\frac{\partial \pi_{\max}}{\partial z_{\max}}\right|_{z_{\max}=0} \leq 0.$

Thus the only possibilities are:

1. Z = 0,
2. $z_i > 0$, $z_{max} = 0$,
3. z_i , $z_{max} > 0$.

Note that: $\frac{1-F^Z(x_{\max})}{Z}$ is a decreasing function of Z, and $\lim_{z \to 0}\frac{1-F^Z(x_{\max})}{Z} = -\log F(x_{\max})$

If $-\log F(x_{\max}) \leq \frac{c}{P}$, then $\left.\frac{\partial \pi_i}{\partial z_i}\right|_{z_i=0} = P\frac{1-F^Z(x_{\max})}{Z} - c \leq -P\log F(x_{\max}) - c \leq 0,$ and $Z = 0$

If $z_{max} > 0$, summing the F.O.C.s yields:

$$\frac{m-1}{m}\ \frac{1-F^Z(x_{\max})}{Z} - \frac{c}{P} = 0$$

In order for $z_{max} = 0$, we must have:

$$\frac{m-2}{m-1}\ \frac{1-F^Z(x_{\max})}{Z} - \frac{1}{(m-1)} F^Z(x_{\max}) \log F(x_{\max}) - \frac{c}{P} = 0 \ \text{ and } \ \frac{m-1}{m}\ \frac{1-F^Z(x_{\max})}{Z} - \frac{c}{P} \leq 0$$

Consider the function:

$$\frac{m-2}{m-1}\ \frac{1-F^Z(x_{\max})}{Z} - \frac{1}{(m-1)} F^Z(x_{\max}) \log F(x_{\max}) - \frac{m-1}{m}\ \frac{1-F^Z(x_{\max})}{Z}$$

$$= \ -\frac{1}{Z(m-1)m}\left[1 - F^Z(x_{\max}) + mF^Z(x_{\max}) \log F^Z(x_{\max})\right]$$

$$= \ -\frac{1}{Z(m-1)m}\left[\psi(F^Z)\right]$$

Thus:

$$\frac{m-2}{m-1}\ \frac{1-F^Z(x_{\max})}{Z} - \frac{1}{(m-1)} F^Z(x_{\max}) \log F(x_{\max}) \ \substack{\geq \\ \leq}\ \frac{m-1}{m}\ \frac{1-F^Z(x_{\max})}{Z} \ \textbf{ as}\text{: } F^Z(x_{\max}) \ \substack{\geq \\ \leq}\ \bar{u}_m$$

At: $F^Z(x_{\max}) \ = \ \bar{u}_m$, $\dfrac{\log \bar{u}_m}{\log F^Z} = 1$ but it is always true that $\dfrac{1}{Z} \log F^Z \ = \log F$

(Note: For notational simplicity we will write $F(x_{\max}) \ = F$)

so, at $F^Z(x_m) \ = \ \bar{u}_m$, then $Z = \dfrac{\log \bar{u}_m}{\log F}$ and $\dfrac{m-1}{m}\ \dfrac{1-F^Z}{Z} = -(m-1)F^Z \log F = -(m-1)\bar{u}_m \log F$

Therefore: If: $\dfrac{c}{P} < -(m-1)\bar{u}_m \log F(x_{\max})$,

then $\dfrac{m-1}{m}\ \dfrac{1-F^Z(x_{\max})}{Z} > \dfrac{m-2}{m-1}\ \dfrac{1-F^Z(x_{\max})}{Z} - \dfrac{F^Z(x_{\max}) \log F(x_{\max})}{m-1}$, and $z_{\max} > 0$.

(see figure below)

If $-\log F > \dfrac{c}{P} > -(m-1)\bar{u}_m \log F$, then we must show $z_{\max} = 0$.

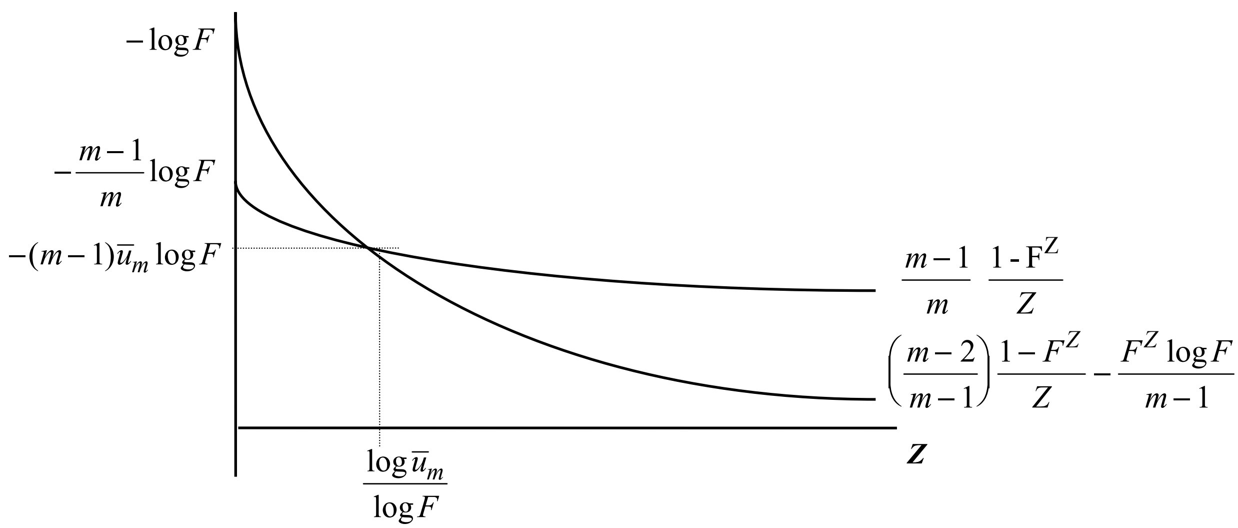
Suppose $z_m > 0$, so $\dfrac{m-1}{m}\ \dfrac{1-F^Z}{Z} = \dfrac{c}{P}$.

Then: $\frac{\partial \pi_i}{\partial z_i} = P\frac{1-F^Z}{Z} - P\frac{z_i}{Z}\left[\frac{1-F^Z}{Z} + F^Z \log F\right] - c$

$$\geq P\frac{1-F^Z}{Z} - \frac{P}{m-1}\left[\frac{1-F^Z}{Z} + F^Z \log F\right] - c \qquad \text{because } z_m > 0$$

$$> P\left(\frac{m-1}{m}\right)\frac{1-F^Z}{Z} - c = 0 \qquad \text{because } F^Z > \overline{u}_m \qquad \textbf{Q.E.D.}$$

Graphically,



In the figure above, $z_{max} = 0$ to the left of the dotted line at $\frac{\log \overline{u}_m}{\log F}$, and to the right $z_{max} > 0$.

On the vertical axis, when $\frac{c}{P} > -(m-1)\overline{u}_m \log F$ we will be to the left of the dotted line on the horizontal axis and $z_{max} = 0$.

How can you determine what x_{max} is correct for finding where $z_{max} = 0$?

We know at that value, $\frac{c}{P} = -(m-1)\overline{u}_m \log F$. Therefore, the appropriate x_{max} is the one that solves:

$$x_{\max} = F^{-1}\left\{e^{\left[-c/P(m-1)\overline{u}_m\right]}\right\}$$

Thus, as an example if $c = 1$, $P = 10$, $m = 2$, and $F(x)$ is uniform [0,1], then if $x_{max} \geq 0.70378$ this implies that $z_{max} = 0$.

LEMMA A3: If $-(m-1)\bar{u}_m \log F(x_{\max}) > \frac{c}{P}$, then π_i is increasing in x_{max}

If $-(m-1)\bar{u}_m \log F(x_{\max}) \leq \frac{c}{P}$, then π_i is nonincreasing in x_{max}

If $-\log F(x_{\max}) < \frac{c}{P}$, then Z = 0 and $\pi_i = 0$.

Proof: If $\frac{c}{P} \geq -\log F(x_{\max})$, then $Z = 0$ and $\pi_i = 0$ from Lemma A2.

So suppose $-(m-1)\bar{u}_m \log F \leq \frac{c}{P} \leq -\log F$, so $z_i > 0$ and $z_{max} = 0$.

$$\pi_i = P\frac{z_i}{Z}(1-F^Z) - cz_i = \frac{P}{m-1}\left[1 - F^Z - \frac{c}{P}Z\right]$$

Thus, $$\frac{d\pi_i}{dF} = \frac{P}{m-1}\left[-ZF^{Z-1} - \left(F^Z \log F + \frac{c}{P}\right)\frac{dZ}{dF}\right] = \frac{-P}{m-1}\left[ZF^{Z-1} + \left(F^Z \log F + \frac{c}{P}\right)\frac{dZ}{dF}\right]$$

Z is determined by: $v = (m-2)(1-F^Z) - ZF^Z \log F - (m-1)\frac{c}{P}Z = 0$

$$\frac{\partial v}{\partial Z} = -(m-2)F^Z \log F - F^Z \log F - ZF^Z(\log F)^2 - (m-1)\frac{c}{P}$$

$$= -(m-1)F^Z \log F - ZF^Z(\log F)^2 - \left[(m-2)\frac{(1-F^Z)}{Z} - F^Z \log F\right]$$

$$= -\left\{(m-2)\left[\frac{(1-F^Z)}{Z} + F^Z \log F\right] + ZF^Z(\log F)^2\right\}$$

$$\frac{\partial v}{\partial F} = -(m-2)ZF^{Z-1} - Z^2F^{Z-1}\log F = -ZF^{Z-1}\left[(m-2) + Z\log F\right]$$

$$\frac{dz}{dF} = -\frac{\frac{\partial v}{\partial F}}{\frac{\partial v}{\partial Z}} = -\frac{ZF^{Z-1}\left[(m-2) + Z\log F\right]}{\left\{(m-2)\left[\frac{(1-F^Z)}{Z} + F^Z \log F\right] + ZF^Z(\log F)^2\right\}}$$

Note that the denominator is positive since $\left[\frac{(1-F^Z)}{Z}+F^Z\log F\right]>0$

Thus, $\frac{d\pi_i}{dF}<0$ IFF $ZF^{Z-1}+\left(F^Z\log F+\frac{c}{P}\right)\frac{dZ}{dF}>0$

IFF $$ZF^{Z-1}-\frac{ZF^{Z-1}\left(F^Z\log F+\frac{c}{P}\right)\left[(m-2)+Z\log F\right]}{\left\{(m-2)\left[\frac{(1-F^Z)}{Z}+F^Z\log F\right]+ZF^Z(\log F)^2\right\}}>0$$

IFF $$(m-2)\left[\frac{(1-F^Z)}{Z}+F^Z\log F\right]+ZF^Z(\log F)^2>\left(F^Z\log F+\frac{c}{P}\right)\left[(m-2)+Z\log F\right]$$

IFF

$$(m-2)\left[\frac{(1-F^Z)}{Z}+F^Z\log F\right]+ZF^Z(\log F)^2$$

$$>\left(F^Z\log F+\frac{m-2}{m-1}\frac{1-F^Z}{Z}-\frac{F^Z\log F}{m-1}\right)\left[(m-2)+Z\log F\right]$$

IFF $$(m-2)\left[\frac{(1-F^Z)}{Z}+F^Z\log F\right]+ZF^Z(\log F)^2>\left(\frac{m-2}{m-1}\right)\left(\frac{1-F^Z}{Z}+F^Z\log F\right)\left[(m-2)+Z\log F\right]$$

IFF $$(m-2)\left[\frac{(1-F^Z)}{Z}+F^Z\log F\right]+ZF^Z(\log F)^2>\left(\frac{m-2}{m-1}\right)\left(\frac{1-F^Z}{Z}+F^Z\log F\right)\left[(m-1)+Z\log F-1\right]$$

IFF $$ZF^Z(\log F)^2>\left(\frac{m-2}{m-1}\right)\left(\frac{1-F^Z}{Z}+F^Z\log F\right)\left[Z\log F-1\right]$$

Which is true. (left hand side > 0 > right hand side, since $\left[\frac{(1-F^Z)}{Z}+F^Z\log F\right]>0$)

(For $m = 2$, the proof is trivial: $\frac{d\pi_i}{dF}=\frac{\partial\pi_i}{\partial F}<0$.)

Thus, we have shown that if $-(m-1)\bar{u}_m\log F(x_{\max})\leq\frac{c}{P}$, then π_i is nonincreasing in x_{max}.

On the other hand, if $\frac{c}{P}<-(m-1)\bar{u}_m\log F$, $z_{\max}>0$ and $\frac{m-1}{m}\frac{1-F^Z}{Z}=\frac{c}{P}$

$$\pi_i = \frac{Pz_i}{Z}(1 - F^Z) - cz_i = Pz_i \left\{ \frac{c}{P} \frac{m}{m-1} \right\} - cz_i = \frac{cz_i}{m-1}$$

$$0 = \frac{\partial \pi_i}{\partial z_{\max}} = P \frac{Z - z_{\max}}{Z} \left[F^Z \log F + \frac{1 - F^Z}{Z} \right] - c = P \frac{(m-1)z_i}{Z} \left[F^Z \log F + \frac{m}{m-1} \frac{c}{P} \right]$$

Thus, $z_i = \frac{c}{P(m-1)} \left[\frac{Z}{\frac{m}{m-1} \frac{c}{P} + F^Z \log F} \right]$

However, using the quotient rule for differentiating, since the denominator is squared we know:

$$\frac{dz_i}{dF} < 0 \quad \text{IFF} \quad \left[\frac{m}{m-1} \frac{c}{P} + F^Z \log F \right] \frac{dZ}{dF} - Z \frac{d}{dF} F^Z \log F < 0$$

$$\text{IFF} \quad \left[\frac{m}{m-1} \frac{c}{P} + F^Z \log F \right] \frac{dZ}{dF} - Z \left[F^Z (\log F)^2 \frac{dZ}{dF} + F^{z-1} \{ Z \log F + 1 \} \right] < 0$$

$$\text{IFF} \quad \left[\frac{m}{m-1} \frac{c}{P} + F^Z \log F - ZF^Z (\log F)^2 \right] \frac{dZ}{dF} - ZF^{z-1} \{ Z \log F + 1 \} < 0$$

Since $1 - F^Z - \frac{m}{m-1} \frac{cZ}{P} = 0$ then by the implicit function rule we have:

$$\frac{dZ}{dF} = -\frac{ZF^{Z-1}}{F^Z \log F + \frac{mc}{(m-1)P}} < 0 \quad \text{since} \quad F^Z \log F + \frac{mc}{(m-1)P} = F^Z \log F + \frac{1 - F^Z}{Z} > 0$$

Therefore, $\frac{dz_i}{dF} < 0$ IFF:

$$-\left[\frac{m}{m-1} \frac{c}{P} + F^Z \log F - ZF^Z (\log F)^2 \right] ZF^{Z-1} - ZF^{z-1} [Z \log F + 1] \left[F^Z \log F + \frac{mc}{(m-1)P} \right] < 0$$

$$\text{IFF} \quad \left[\frac{m}{m-1} \frac{c}{P} + F^Z \log F - ZF^Z (\log F)^2 \right] ZF^{Z-1} + ZF^{z-1} [Z \log F + 1] \left[F^Z \log F + \frac{mc}{(m-1)P} \right] > 0$$

$$\text{IFF} \quad \left[\frac{m}{m-1} \frac{c}{P} + F^Z \log F - ZF^Z (\log F)^2 \right] + [Z \log F + 1] \left[F^Z \log F + \frac{mc}{(m-1)P} \right] > 0$$

IFF $\left[Z\log F+2\right]\left[F^{Z}\log F+\dfrac{mc}{(m-1)P}\right]>ZF^{Z}(\log F)^{2}$

IFF $\left[Z\log F+2\right]\left[F^{Z}\log F+\dfrac{1-F^{Z}}{Z}\right]>ZF^{Z}(\log F)^{2}$

IFF $\left[Z\log F+2\right]\left[ZF^{Z}\log F+1-F^{Z}\right]>Z^{2}F^{Z}(\log F)^{2}$

IFF $\left[\log F^{Z}+2\right]\left[F^{Z}\log F^{Z}+1-F^{Z}\right]>F^{Z}(\log F^{Z})^{2}$ (since mathematically $x\log F=\log F^{x}$)

IFF $\left[\log F^{Z}+2\right]\left[1-F^{Z}\right]+2F^{Z}\log F^{Z}>0$

Let $u=F^{Z}$, and $\widetilde{\varphi}(u)=\left[\log u+2\right]\left[1-u\right]+2u\log u$

$$\widetilde{\varphi}'(u)=\frac{1-u}{u}-\left[\log u+2\right]+2u\log u+2=\frac{1}{u}-1+\log u>0$$

{since $1-u+u\log u$ is zero at $u=1$ and is decreasing for $u\in(0,1)$ }

Therefore, when $u\in(0,1)$ then $\widetilde{\varphi}(u)<0$ because $\widetilde{\varphi}(1)=0$ and $\widetilde{\varphi}'(u)>0$.

Thus, (*note we are reversing the inequalities now*) a sufficient condition for $\dfrac{d\,\pi_i}{d\,F}>0$ is $\dfrac{d\,z_i}{d\,F}>0$, or

$\widetilde{\varphi}(u)<0$ where $u=F^{Z}$. Moreover, $\widetilde{\varphi}(u)<0$ need only hold at the upper limit of u, defined by:

$$\frac{m-1}{m}\ \frac{1-F^{Z}}{Z}=\frac{c}{P}<-(m-1)\overline{u}_m\log F$$

IFF $1-F^{Z}+m\overline{u}_m\log F^{Z}<0$

IFF $1-u+m\overline{u}_m\log u<0$ {Note the LHS is increasing in u and is zero at $u=\overline{u}_m$}

IFF $u<\overline{u}_m$

Thus, a *sufficient* condition for $\dfrac{d\,\pi_i}{d\,F}>0$ is:

$\left[\log\overline{u}_m+2\right]\left[1-\overline{u}_m\right]+2\overline{u}_m\log\overline{u}_m<0$ but $\overline{u}_m\log\overline{u}_m=-\dfrac{1-\overline{u}_m}{m}$

So, a sufficient condition for $\dfrac{d\,\pi_i}{d\,F}>0$ is:

$\left[\log\overline{u}_m+2\right]\left[1-\overline{u}_m\right]-\dfrac{2(1-\overline{u}_m)}{m}<0$ OR $2\left(\dfrac{m-1}{m}\right)+\log\overline{u}_m<0$

When $m = 2$, $\bar{u}_m = 0.284668$ therefore: $2\left(\frac{m-1}{m}\right) + \log \bar{u}_m = -0.25643 < 0$

When $m = 3$, $\bar{u}_m = 0.148999$ therefore: $2\left(\frac{m-1}{m}\right) + \log \bar{u}_m = -0.570482 < 0$

When $m \geq 4$ this is also true because $\bar{u}_m$ is defined as the solution to: $1 - u + mu \log u = 0$. So, checking $u = e^{-2}$ we get: $1 - e^{-2} + me^{-2}(-2) = 1 - e^{-2}[1 + 2m] < 0$ whenever $m \geq 4$.

Q.E.D.

Theorem 4: The uniform-price auction is inefficient for selecting contestants when firms differ in their initial starting positions.

Proof: Previously we showed that if firms are not allowed to conduct research following entry then the only possible pure-strategy bidding equilibrium is for all firms to bid 0, which is not an equilibrium. When firms are allowed to conduct research following entry, LEMMA 4 tells us that Z=0 IFF

$F(x_{\max}) \geq e^{-c/P}$. This implies that for all tournaments where $c > 0$ and $P < \infty$, then $F(x_{max}) < 1$. In other words, the firms with the best possible endowments (i.e., innovations greater than this critical value of x_{max}) will NEVER do additional research following entry. If the sponsor conducts a uniform-price entry auction, the expected profits of one of these firms holding x which bids as if it holds r but never does additional research is:

$$u(r,x) = P \int_0^{\min\{r,x\}} \left(\frac{F(x) - F(y)}{1 - F(y)} \right)^{m-1} f_{m:n-1}(y)dy - \int_0^r B(y) f_{m:n-1}(y)dy$$

Using Lemma A1 and taking the derivative with respect to r, evaluated at $r = x$ leaves us with:

$$u_1(r,x) \Big|_{r=x} = 0 \qquad \Rightarrow \qquad B(x) = 0$$

However, this means that the only possible pure-strategy equilibrium bid for firms with the largest endowments of x is to always bid zero regardless of whether additional research is allowed by the sponsor following entry. But this obviously cannot be an equilibrium bidding strategy -- so there is no efficient equilibrium for the uniform-price entry auction if firms differ in their initial starting positions. **Q.E.D.**

Section V

THEOREM 5: If $\exists\ \tilde{w} \in [0, \overline{w}]$ such that $\forall$ *w* < $\tilde{w}$, $\pi(w,w) \geq \pi(\tilde{w},\tilde{w})$, a symmetric, pure-strategy bidding equilibrium does not exist for the discriminatory-price or uniform-price entry auction.

Proof: If the sponsor conducts a uniform-price entry auction and contestant *i* deviates from the bidding equilibrium by bidding as type *w** his *ex ante* expected profit from the tournament will be:

$$\psi_i = \int_0^{w^*} [\pi_i(w_i, w_{m:n-1}) - B_{m+1}(w_{m:n-1})] h_{m:n-1}(w_{m:n-1}) dw_{m:n-1}$$

The first order conditions for this expression suggest the only possible equilibrium for this uniform-price entry auction is the bidding function: $B_{m+1}(w_i) = \pi_i(w_i, w_i)$. This implies that firms must bid their expected profits given entry in the tournament, assuming they will have the smallest endowment of *w* of all the entrants. But the condition, $\exists\ \tilde{w} \in [0, \overline{w}]$ such that $\forall\ w < \tilde{w}$, $\pi(w,w) \geq \pi(\tilde{w},\tilde{w})$, implies this bid is either decreasing for *w* sufficiently close to $\tilde{w}$, or is constant for all $w < \tilde{w}$. Therefore a symmetric, increasing, pure-strategy bidding equilibrium cannot exist. In the discriminatory price auction, the first order conditions give:

$B_1'(w_i) = [\pi_i(w_i, w_i) - B_1(w_i)]\dfrac{h(w_i)}{H(w_i)}$. Using $B_1(0) = 0$ we can get $B_1(w_i) = \int_0^{w_i} \pi(y,y)\dfrac{h(y)}{H(w_i)}dy$

Therefore, $B_1'(w_i) = \int_0^{w_i} [\pi_i(w_i, w_i) - \pi(y,y)\dfrac{h(y)}{H(w_i)}]dy \dfrac{h(w_i)}{H(w_i)}$ implying once again that either

$B_1'(w_i) = 0 \ \ \forall\ w < \tilde{w}$, or $B_1'(w_i) < 0$ for *w* sufficiently close to $\tilde{w}$. **Q.E.D.**

Now, using expectation notation we will make the connection between our findings in Theorem 5 and our results from Section III and Section IV.

Let the *w* be independently distributed with cdf *H(w)* and density *h(w)*. There are *m* firms that gain entry and by assumption we have it that w_i is the smallest *w* of all entrants, implying that all entrants can be said to have been drawn independently from the modified distribution of $\dfrac{H(y) - H(w_i)}{1 - H(w_i)}$ with a corresponding density of $\frac{h(y)}{1-H(w_i)}$. Let $h_{m:n-1}(w_{m:n-1})$ be the density of the m+1st highest bidding firm.

If firm i has endowment w_i, but bids $B_{m+1}(\hat{w})$ in a uniform-price auction when its equilibrium bid is $B_{m+1}(w_i)$ then firm i obtains profits of:

$$\psi_i = \int_0^{\hat{w}} \left[E\left\{\pi_i(w_i;w_{-i}) | w_{-i} \geq w_{m:n-1}\right\} - B_{m+1}(w_{m:n-1}) \right] h_{m:n-1}(w_{m:n-1}) dw_{m:n-1}$$

First order conditions give: $B_{m+1}(w_i) = E\left\{\pi_i(w_i;w_{-i}) | w_{-i} \geq w_i\right\}$

which corresponds to our previous findings in Theorem 5 for the uniform-price auction.

Suppose $B_1(w_i)$ is firm i's bidding function for the discriminatory-price auction. Then profits, if it bids $B_1(\hat{w})$ are:

$$\phi_i = \int_0^{\hat{w}} \left[E\left\{\pi_i(w_i;w_{-i}) | w_{-i} \geq w_{m:n-1}\right\} - B_1(\hat{w}) \right] h_{m:n-1}(w_{m:n-1}) dw_{m:n-1}$$

First order conditions give:

$$\frac{\partial B_1(w_i)}{\partial w_i} = \frac{h(w_i)}{H(w_i)}\left[E\left\{\pi_i(w_i;w_{-i}) | w_{-i} \geq w_i\right\} - B_1(w_i) \right] = \frac{h(w_i)}{H(w_i)}\left[B_{m+1}(w_i) - B_1(w_i) \right]$$

Since reserve ≥ 0, for some K ≥ 0 it is true that: $B_1(w_i) = \frac{1}{H(w_i)}\left[K + \int_{\underline{w}}^{w_i} B_{m+1}(y) h(y) dy \right]$

Thus, $\frac{\partial B_1(w_i)}{\partial w_i} = \frac{h(w_i)}{H(w_i)}\left[-\frac{K}{H(w_i)} + \int_{\underline{w}}^{w_i} \left[B_{m+1}(w_i) - B_{m+1}(y) \right] \frac{h(y)}{H(w_i)} dy \right]$.

We need to find an expression for: $\frac{\partial \mathrm{E}\left\{\pi_i(w_i;w_{-i}) | w_{-i} \geq w_i\right\}}{\partial w_i}$

To begin, we can write:

$$E\left[\pi(w_i;w_k)-\pi(w_i;w_i)|w_k \geq w_i\right] = \int_{w_i}^{\infty}\left[\pi(w_i;w_k)-\pi(w_i;w_i)\right]\frac{h_k(w_k)}{1-H_k(w_i)}dw_k$$

$$= -\left[\pi(w_i;w_k)-\pi(w_i;w_i)\right]\frac{1-H_k(w_k)}{1-H_k(w_i)}\Bigg|_{w_k=w_i}^{\infty} + \int_{w_i}^{\infty}\frac{\partial\pi(w_i;w_k)}{\partial w_k}\frac{1-H_k(w_k)}{1-H_k(w_i)}dw_k$$

but when $w_k = \infty$ then $[1\text{-}H(w_k)] = 0$, therefore the first term directly above goes to zero and we can rewrite the second term to leave us with:

(%) $$E\left[\pi(w_i;w_k)-\pi(w_i;w_i)|w_k \geq w_i\right] = E\left[\frac{\partial\pi(w_i;w_k)}{\partial w_k}\frac{1-H_k(w_k)}{h_k(w_k)}\Bigg| w_k \geq w_i\right]$$

However: $\dfrac{\partial \mathrm{E}\{\pi_i(w_i;w_{-i})|w_{-i} \geq w_i\}}{\partial w_i} =$

$$\mathrm{E}\left\{\frac{\partial\pi_i(w_i;w_{-i})}{\partial w_i}\right\} - \sum_{K\neq i}\int_{w_i}^{\infty}\dots\int_{w_i}^{\infty}\pi_i(w_i;w_i,w_{-i-k})\prod_{j\neq i,k}\frac{h_j(w_j)dw_j}{1-H_j(w_i)}\frac{h_k(w_i)}{1-H_k(w_i)}$$
$$+\sum_{K\neq i}\frac{h_k(w_i)}{1-H_k(w_i)}\int_{w_i}^{\infty}\dots\int_{w_i}^{\infty}\pi_i(w_i;w_{-i})\prod_{j\neq i}\frac{h_j(w_j)dw_j}{1-H_j(w_i)}$$

$$= \mathrm{E}\left\{\frac{\partial\pi_i(w_i;w_{-i})}{\partial w_i}\right\} + \sum_{K\neq i}\frac{h_k(w_i)}{1-H_k(w_i)}\int_{w_i}^{\infty}\dots\int_{w_i}^{\infty}\left[\pi_i(w_i;w_{-i})-\pi_i(w_i;w_i,w_{-i-k})\right]\prod_{j\neq i}\frac{h_j(w_j)dw_j}{1-H_j(w_i)}$$

We can now substitute (%) into this expression to get:

$$\frac{\partial \mathrm{E}\{\pi_i(w_i;w_{-i})|w_{-i} \geq w_i\}}{\partial w_i} = \mathrm{E}\left\{\frac{\partial\pi_i(w_i;w_{-i})}{\partial w_i} + \sum_{k\neq i}\frac{\partial\pi(w_i;w_{-i})}{\partial w_k}\frac{h_k(w_i)}{1-H_k(w_i)}\frac{1-H_k(w_k)}{h_k(w_k)}\Bigg| w_k \geq w_i\right\}$$

But this equality just simplifies to:

$$\frac{\partial \mathrm{E}\{\pi_i(w_i;w_{-i})|w_{-i} \geq w_i\}}{\partial w_i} = \mathrm{E}\left\{\sum_{j=1}^{m}\frac{\partial\pi(w_i;w_{-i})}{\partial w_j}\frac{h_j(w_i)}{1-H_j(w_i)}\frac{1-H_j(w_j)}{h_j(w_j)}\Bigg| w_j \geq w_i\right\}$$

Moreover, when all competitors get their attribute endowments from the same distribution, we can further simplify the expression to be:

$$\frac{\partial \mathrm{E}\{\pi_i(w_i;w_{-i})|w_{-i}\geq w_i\}}{\partial w_i}=\frac{h(w_i)}{1-H(w_i)}\left[\mathrm{E}\left\{\sum_{j=1}^{m}\frac{\partial \pi(w_i;w_{-i})}{\partial w_j}\frac{1-H(w_j)}{h(w_j)}\middle| w_j\geq w_i\right\}\right]$$

from earlier we know that: $B_{m+1}(w_i)=E\{\pi_i(w_i;w_{-i})|w_{-i}\geq w_i\}$

Therefore, we can see that in the general case:

$$(\$)\qquad \frac{\partial B_{m+1}(w_i)}{\partial w_i}=\frac{h(w_i)}{1-H(w_i)}\left[\mathrm{E}\left\{\sum_{j=1}^{m}\frac{\partial \pi(w_i;w_{-i})}{\partial w_j}\frac{1-H(w_j)}{h(w_j)}\middle| w_j\geq w_i\right\}\right]$$

Thus, if $\sum_{j=1}^{m}\frac{\partial \pi(w_i;w_{-i})}{\partial w_j}\frac{1-H(w_j)}{h(w_j)}\geq 0$, a symmetric equilibrium exists.

However, in Theorem 5 of Section IV we showed that when *w* represents initial starting positions then there is always some range of *w* such that $\frac{\partial \pi(w_i;w_{-i})}{\partial w_j}<0$ which means there is no symmetric equilibrium which was the subject of Theorem 4.

In Section III we saw that uniform-price auctions may work for differences in cost between firms as long as a specific hazard rate condition is met by the cost distribution. When contestants differ in their costs, the sponsor wishes to induce the *lowest* cost firms to enter so we must treat *w* as the inverse of costs, *c*, and condition ($) becomes transformed into:

$$(\$c)\qquad \frac{\partial B_{m+1}(c_i)}{\partial c_i}=\frac{h(c_i)}{H(c_i)}\left[\mathrm{E}\left\{\sum_{j=1}^{m}\frac{\partial \pi(c_i;c_{-i})}{\partial c_j}\frac{H(c_j)}{h(c_j)}\middle| c_{-i}\leq c_i\right\}\right]$$

Therefore, if $E\left\{\sum_{j=1}^{m}\frac{\partial \pi(c_i;c_{-i})}{\partial c_j}\frac{H(c_j)}{h(c_j)}\right\}<0$, a symmetric bidding equilibrium exists and the (m+1)st price auction is efficient for selecting the lowest cost contestants. But this in turn is true if $\frac{ch(c)}{H(c)}$ is decreasing in *c*, which was our requirement from Section III for showing an efficient bidding equilibrium.

THEOREM 6: The contestant-selection auction is an efficient mechanism for selecting the best-qualified contestants to participate in a tournament.

Proof: If contestant i holds endowment w and bids as if holding $\hat{w}$ his expected profit from the tournament will be:

$$u = \int_0^{\hat{w}} \pi(w,y) h_{m:n-1}(y)dy + K\int_0^{\hat{w}} h_{m:n-1}(y)dy - B(\hat{w})$$

$$\frac{\partial u}{\partial \hat{w}} = \pi(w,\hat{w})h_{m:n-1}(\hat{w}) + Kh_{m:n-1}(\hat{w}) - B'(\hat{w}) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial \hat{w}\partial w} = \frac{\partial \pi(w,\hat{w})}{\partial w} h_{m:n-1}(\hat{w}) \geq 0$$

$$\left.\frac{\partial u}{\partial w}\right|_{\hat{w}=w} = \pi(w,w)h_{m:n-1}(w) + Kh_{m:n-1}(w) - B'(w)$$

Setting $\left.\frac{\partial u}{\partial w}\right|_{\hat{w}=w} = 0$ we have: $B'(w) = \pi(w,w)h_{m:n-1}(w) + Kh_{m:n-1}(w)$

Therefore, if an initial value is imposed of: *B(0)=0* we arrive at the unique solution of:

$$B(w) = \int_0^{w} \pi(y,y) h_{m:n-1}(y)dy + K\int_0^{w} h_{m:n-1}(y)dy$$

Since $\pi(w,w) \geq 0 \;\; \forall\, w$, the contestant-selection auction always has a symmetric, pure-strategy bidding equilibrium which is strictly increasing in the winning attribute *w*. **Q.E.D.**

THEOREM 7: The expected cost of implementing a tournament using the contestant-selection auction is independent of the interim prize, *K*, and equivalent to the expected cost of implementing a tournament using a uniform-price entry auction whenever the uniform-price auction is efficient.

Proof: The *ex ante* expected bid of an arbitrary contestant which draws attribute *w* from *H(w)* is:

$$\int_0^{\overline{w}} B(w)h(w)dw$$

Integrating by parts, let us define: $u = B(w)$ and $v = -[1-H(w)]dw$; $du = B'(w)dw$ and $dv = h(w)dw$

Therefore: $\int_0^{\overline{w}} B(w)h(w)dw = -B(w)[1-H(w)\Big|_0^{\overline{w}} + \int_0^{\overline{w}} B'(w)[1-H(w)]dw$

The first term on the right-hand side of the equality is equal to zero, so we have:

$$\int_0^{\overline{w}} B(w)h(w)dw = \int_0^{\overline{w}} B'(w)[1-H(w)]dw = \int_0^{\overline{w}} \{\pi(w,w)+K\}[1-H(w)]h_{m:n-1}(w)dw$$

However, $h_{m:n-1}(w) = \dfrac{(n-1)!}{(n-1-m)!(m-1)!}[H(w)]^{n-1-m}[1-H(w)]^{m-1}h(w)$

and $h_{m+1:n}(w) = \dfrac{n!}{(n-1-m)!m!}[H(w)]^{n-m-1}[1-H(w)]^{m}h(w)$

therefore,

$$\frac{m}{n}h_{m+1:n}(w) = [1 - H(w)]h_{m:n}(w)$$

Substituting into the expression above we see that the *ex ante* expected bid is:

$$\int_0^{\overline{w}} B(w)h(w)dw = \frac{m}{n}\int_0^{\overline{w}} \{\pi(w,w)+K\}h_{m+1:n}(w)dw = \frac{m}{n}\int_0^{\overline{w}} \pi(w,w)h_{m+1:n}(w)dw + \frac{mK}{n}$$

Since n firms submit bids, the total expected sponsor take from all n bidders is:

$$m\int_0^{\overline{w}} \pi(w,w)h_{m+1:n}(w)dw + mK$$

However, the sponsor must pay out mK to the m entrants, plus P to the overall winner.

Therefore, the sponsor's total cost of conducting the tournament is:

$$P - m\int_0^{\overline{w}} \pi(w,w)h_{m+1:n}(w)dw$$

which is independent of the sponsor's choice of K. Moreover, we know that if a pure-strategy equilibrium bidding function exists for a uniform-price entry auction (e.g., when contestants differ in their cost of effort and the hazard rate condition is met), a contestant holding endowment *w* would submit an equilibrium entry bid of $\pi(w,w)$. Therefore, the expected payment by each of the *m* entrants in a uniform-price entry auction tournament is: $\int_0^{\overline{w}} \pi(w,w) h_{m+1:n}(w) dw$ so the cost of implementing a tournament using a uniform-price entry auction is: $P - m\int_0^{\overline{w}} \pi(w,w) h_{m+1:n}(w) dw$ which is equivalent to the cost of implementing the contestant-selection auction.

Q.E.D.